Math 6345 – AODE's

Given $\frac{d\bar{x}}{dt} = A\bar{x}$

sol^n $\bar{x} = \Phi(t)\,\bar{c}$ where Φ - fundamental matrix

if $\bar{x}(0) = \bar{x}_0$

then $\Phi(0)\,\bar{c} = \bar{x}_0 \Rightarrow \bar{c} = \Phi^{-1}(0)\,\bar{x}_0$

so $\bar{x} = \Phi(t)\,\Phi^{-1}(0)\,\bar{x}_0$

Also $\bar{x} = e^{At}\,\bar{x}_0$

so $e^{At} = \Phi(t)\,\Phi^{-1}(0)$

e^{At} – Matrix Exponential

Ex $\quad \dfrac{d\bar{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$

Eigenvalue - Eigenvector Problem

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 4\lambda + 3 = 0 \qquad (\lambda-1)(\lambda-3) = 0$

$\lambda = 1, 3$

so $\lambda = 1 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \bar{e} = \bar{0}$ so $\bar{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so $\quad \bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$

$\lambda = 3 \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \bar{e} = \bar{0}$ so $\bar{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so $\quad \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$

$$\bar{x} = \begin{pmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{pmatrix} \bar{c}$$

↑ this is $\Phi(t)$

so $\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\det \Phi(0) = 2$

$$\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

so $e^{At} = \Phi(t)\,\Phi^{-1}(0)$

$$= \frac{1}{2}\begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^t + e^{3t}}{2} & \frac{-e^t + e^{3t}}{2} \\ \frac{-e^t + e^{3t}}{2} & \frac{e^t + e^{3t}}{2} \end{pmatrix}$$

note that $e^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

could we obtain this from the def^n

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

so $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$$A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t + \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \frac{t^3}{3!}$$

what is $1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} \cdots$

& $t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \cdots$

} who knows ?

Well we do because we know $e^{At} = \Phi(t)\Phi^{-1}(0)$

However we note that

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so could we calculate

$$e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} t} \cdot e^{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} t}$$

we first note

$$e^{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^2 \frac{t^2}{2!} \cdots$$

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & d^n \end{pmatrix}$$

so

$$e^{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t} = \begin{pmatrix} 1 + at + \frac{a^2 t^2}{2!} \cdots & 0 \\ 0 & 1 + dt + \frac{d^2 t^2}{2!} \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{at} & 0 \\ 0 & e^{dt} \end{pmatrix}$$

so for $e^{\begin{pmatrix}1&0\\0&1\end{pmatrix}t} = \begin{pmatrix}e^t & 0\\0 & e^t\end{pmatrix}$

For $e^{\begin{pmatrix}1&1\\1&1\end{pmatrix}t}$ we need $\begin{pmatrix}1&1\\1&1\end{pmatrix}^n$

so $\begin{pmatrix}1&1\\1&1\end{pmatrix}^2 = \begin{pmatrix}2&2\\2&2\end{pmatrix}$ $\begin{pmatrix}1&1\\1&1\end{pmatrix}^3 = \begin{pmatrix}4&4\\4&4\end{pmatrix}$ etc.

so
$$e^{\begin{pmatrix}1&1\\1&1\end{pmatrix}t} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}1&1\\1&1\end{pmatrix}t + \begin{pmatrix}2&2\\2&2\end{pmatrix}\frac{t^2}{2!}\cdots$$

so

$$1+t+\frac{2t^2}{2!}+\frac{4t^3}{3!} = 1+\frac{1}{2}\left(2t+\frac{4t^2}{2!}+\frac{8t^3}{3!}\cdots\right)$$

$$= 1+\frac{1}{2}(e^{2t}-1) = \frac{1+e^{2t}}{2}$$

$$t+\frac{2t^2}{2!}\cdots = \frac{1}{2}\left(2t+\frac{4t^2}{2!}+\frac{8t^3}{3!}\cdots\right)$$

$$= \frac{1}{2}(e^{2t}-1)$$

so $$e^{\begin{pmatrix}1&1\\1&1\end{pmatrix}t} = \begin{pmatrix}\frac{1+e^{2t}}{2} & \frac{-1+e^{2t}}{2}\\ \frac{-1+e^{2t}}{2} & \frac{1+e^{2t}}{2}\end{pmatrix}$$

so $$e^{\begin{pmatrix}2&1\\1&2\end{pmatrix}t} = \begin{pmatrix}e^t & 0\\ 0 & e^t\end{pmatrix}\begin{pmatrix}\frac{1+e^{2t}}{2} & \frac{-1+e^{2t}}{2}\\ \frac{-1+e^{2t}}{2} & \frac{1+e^{2t}}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{e^t+e^{3t}}{2} & \frac{-e^t+e^{3t}}{2}\\ \frac{-e^t+e^{3t}}{2} & \frac{e^t+e^{3t}}{2}\end{pmatrix}$$

as before but will this always work?

$$e^{(A+B)t} = e^{At}\cdot e^{Bt} \ ?$$

Better

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now
$$e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$

for $e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t}$ if $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so A^n is either $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ they alternate

so
$$e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{t^3}{3!} \cdots$$

$$= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \cdots & t + \frac{t^3}{3!} + \frac{t^5}{5!} \cdots \\ t + \frac{t^3}{3!} + \frac{t^5}{5!} \cdots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{pmatrix}$$

Now $\quad e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \cdots$

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \cdots$$

so $\quad \frac{e^{t} + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} \cdots$

$$\frac{e^{t} - e^{-t}}{2} = t + \frac{t^3}{3!} + \frac{t^5}{5!}$$

so $\quad e^{\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}t} = \begin{pmatrix} \frac{e^{t}+e^{-t}}{2} & \frac{e^{t}-e^{-t}}{2} \\ \frac{e^{t}-e^{-t}}{2} & \frac{e^{t}+e^{-t}}{2} \end{pmatrix}$

$$e^{\begin{pmatrix}2 & 1 \\ 1 & 2\end{pmatrix}t} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{e^{t}+e^{-t}}{2} & \frac{e^{t}-e^{-t}}{2} \\ \frac{e^{t}-e^{-t}}{2} & \frac{e^{t}+e^{t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{3t}+e^{t}}{2} & \frac{e^{3t}-e^{+t}}{2} \\ \frac{e^{3t}-e^{t}}{2} & \frac{e^{3t}+e^{t}}{2} \end{pmatrix}$$ as before!