Math 1497 - Calc 2

Trig sub

$\int \sqrt{a^2 - x^2}\, dx$ - $x = a \sin\theta$ $dx = a\cos\theta\, d\theta$

$\int \sqrt{a^2 + x^2}\, dx$ - $x = a\tan\theta$ $dx = a\sec^2\theta\, d\theta$

$\int \sqrt{x^2 - a^2}\, dx$ - $x = a\sec\theta$ $dx = a\sec\theta\tan\theta\, d\theta$

Now we turn our attention to definite integrals

Ex 1 $\displaystyle\int_0^2 \frac{dx}{(x^2+4)^{3/2}}$

Here we let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

so $x^2 + 4 = 4\tan^2\theta + 4 = 4(1 + \tan^2\theta)$

$= 4\sec^2\theta$

so $(x^2+4)^{3/2} = \left[(2\sec\theta)^2\right]^{3/2} = (2\sec\theta)^3$

Limits $x = 0 \quad 2\tan\theta = 0 \quad \tan\theta = 0$

$\theta = 0$

$x = 2 \quad 2\tan\theta = 2 \quad \tan\theta = 1$

$\theta = \pi/4$

So $$\int_0^{\pi/4} \frac{2\sec^2\theta \, d\theta}{8\sec^3\theta} = \frac{1}{4}\int_0^{\pi/4} \cos\theta \, d\theta = \frac{\sin\theta}{4}\Big|_0^{\pi/4} = \frac{\sqrt{2}}{8}$$

Ex 2 $$\int_{\sqrt{2}}^{2} \frac{dx}{(x^2-1)^{3/2}}$$

Here $x = \sec\theta$, $dx = \sec\theta \tan\theta \, d\theta$

$x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$

$(x^2-1)^{3/2} = \tan^3\theta$

Limits $x = \sqrt{2} \quad \sec\theta = \sqrt{2} \quad \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4$

$x = 2 \quad \sec\theta = 2 \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = \pi/3$

New $\int$ $$\int_{\pi/4}^{\pi/3} \frac{\sec\theta\tan\theta}{\tan^3\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\sec\theta\,d\theta}{\tan^2\theta}$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \cot\theta\csc\theta\,d\theta = -\csc\theta\Big|_{\pi/4}^{\pi/3}$$

$$= -\left(\csc\pi/3 - \csc\pi/4\right) = -\left(\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{2}}\right)$$

Ex 3 $$\int_0^{3/2} \frac{x^2\,dx}{\sqrt{9-x^2}}$$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$x=0 \quad 3\sin\theta = 0$

$\theta = 0$

$x = 3/2 \quad 3\sin\theta = 3/2$

$\sin\theta = \frac{1}{2} \quad \theta = \pi/6$

$$\int_0^{\pi/6} \frac{9\sin^2\theta\,\cancel{3\cos\theta}}{\cancel{3\cos\theta}}\,d\theta$$

$$9\int_0^{\pi/6} \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/6} = \frac{9}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

ex 4 $\displaystyle\int_1^{\sqrt{3}} \frac{x^2}{(x^2+1)^{5/2}}\,dx$ let $x = \tan\theta$ $dx = \sec^2\theta\,d\theta$

$1+x^2 = 1+\tan^2\theta = \sec^2\theta$

so $(1+x^2)^{5/2} = \sec^5\theta$

limits $x=1$ $\tan\theta = 1 \Rightarrow \theta = \pi/4$

$x=\sqrt{3}$ $\tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$$\int_{\pi/4}^{\pi/3} \frac{\tan^2\theta}{\sec^5\theta}\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3} \frac{\tan^2\theta\,d\theta}{\sec^3\theta} = \int_{\pi/4}^{\pi/3} \frac{\sin^2\theta}{\cos^2\theta}\cos^3\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \sin^2\theta\cos\theta\,d\theta$$

let $u = \sin\theta$ $du = \cos\theta\,d\theta$ $u = \sin\pi/4 = \frac{\sqrt{2}}{2}$

$u = \sin\pi/3 = \sqrt{3}/2$

$$\int_{\frac{\sqrt{2}}{2}}^{\sqrt{3}/2} u^2\,du = \left.\frac{u^3}{3}\right|_{\frac{\sqrt{2}}{2}}^{\sqrt{3}/2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^2$$