Math 2471 – Calc 3

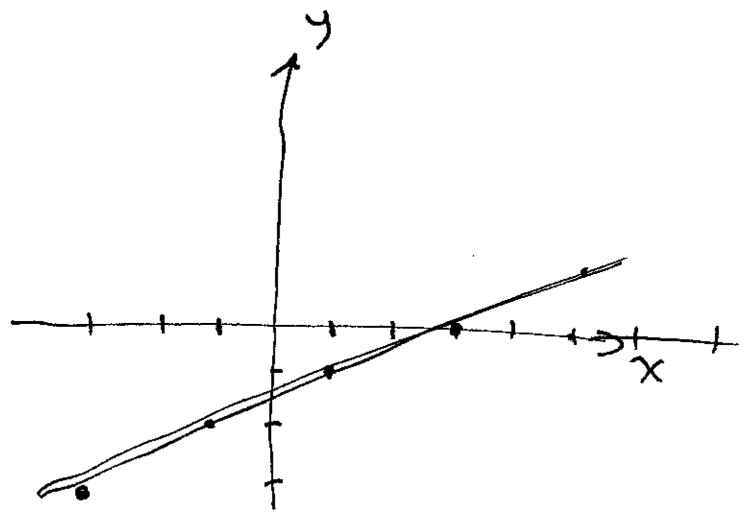
Parametric Eq^n's

In Calc 2 we introduced parametric eq^n's

ex $x = 2t+1$, $y = t-1$, t parameter

Table of Values

t	x	y
-2	-3	-3
-1	-1	-2
0	1	-1
1	3	0
2	5	1
3	7	2



a straight line

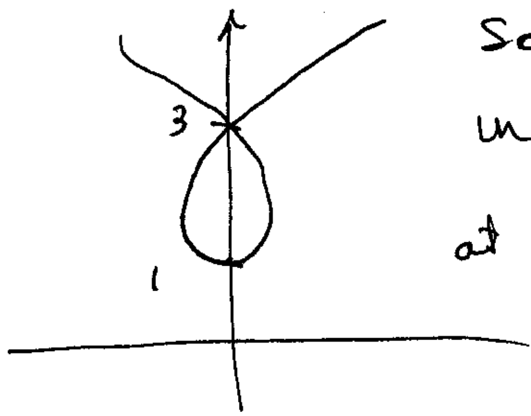
also if we eliminate t, so $x-1 = 2t$

$$t = \frac{x-1}{2} \quad \text{so} \quad y = t-1 = \frac{x-1}{2} - 1 = \frac{x}{2} - 3/2$$

a $y = \frac{x}{2} - 3/2 \leftarrow$ eq^n straight line.

ex 2 consider

$$x = t^3 - 2t, \quad y = t^2 + 1 \quad -2 \leq t \leq 2$$



So where does this curve intersect itself?

at $x = 0, y = 3$

so set $y = 3 \Rightarrow t^2 + 1 = 3 \quad t^2 = 2 \quad t = \pm\sqrt{2}$

check at $t = -\sqrt{2}$ $\quad x = (-\sqrt{2})^3 - 2(-\sqrt{2})$

$$= -2\sqrt{2} + 2\sqrt{2} = 0$$

$t = \sqrt{2}$ same.

Derivatives

we can calculate

$$\frac{dx}{dt} = 3t^2 - 2, \quad \frac{dy}{dt} = 2t$$

So what about $\frac{dy}{dx} = ?$

Simple $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$$= \frac{2t}{3t^2 - 2}$$

So slope of tangent a point of self intersection

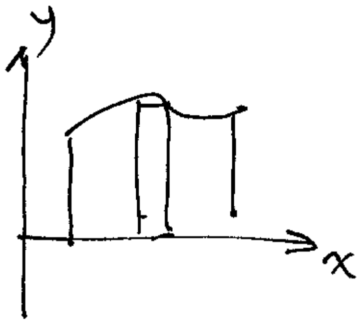
$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{2}} = \frac{2(-\sqrt{2})}{3(-\sqrt{2})^2 - 2} = \frac{-2\sqrt{2}}{4} = -\frac{\sqrt{2}}{2}$$

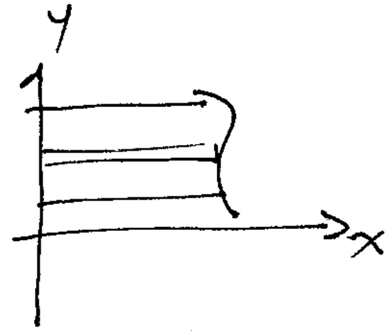
$$\left.\frac{dy}{dt}\right|_{t=+\sqrt{2}} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

tangents $y - 3 = -\frac{\sqrt{2}}{2}(x - 0)$

$$y - 3 = \frac{\sqrt{2}}{2}(x - 0)$$

With parametric eq^n's we can also find area





$$\int y\,dx \qquad\qquad \int x\,dy$$

$$= \int_{t_1}^{t_2} y(t)\,\frac{dx}{dt}\,dt \qquad\qquad \int_{t_3}^{t_4} x(t)\,\frac{dy}{dt}\,dt$$

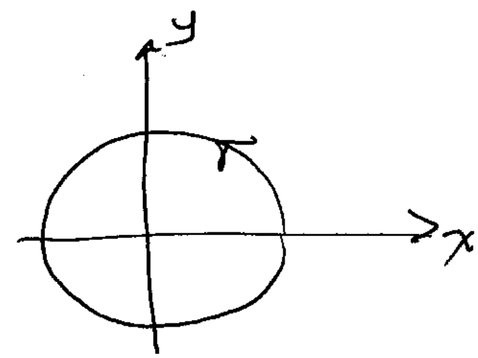
but we must be careful that

$$y,\ \frac{dx}{dt} \geq 0 \quad \& \quad x(t),\ \frac{dy}{dt} \geq 0$$

Another application is arc length

$$s(t) = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

ex $x = 2\cos t$, $y = 2\sin t$ $t: 0 \to 2\pi$



$\frac{dx}{dt} = -2\sin t$

$\frac{dy}{dt} = 2\cos t$

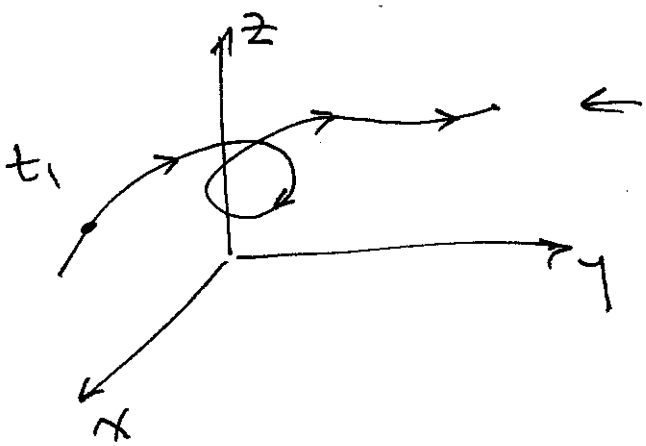
$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_0^{2\pi} \sqrt{4\sin^2 t + 4\cos^2 t}\, dt = \int_0^{2\pi} \sqrt{4}\, dt = 2\int_0^{2\pi} dt$$

$$= 2t\Big|_0^{2\pi} = 4\pi$$

How about 3-D

$x = f(t)$, $y = g(t)$ $z = h(t)$

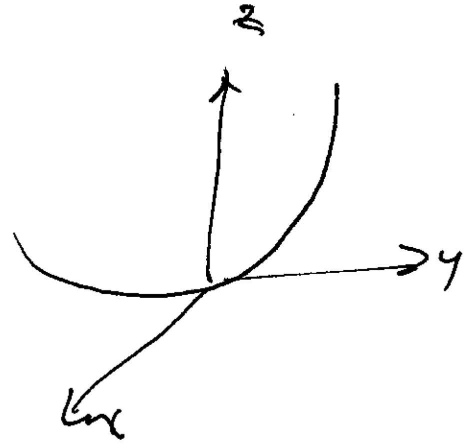


← a 3-D space curve

then arc length

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Ex $\quad x = t^2 + 3t,\ y = \frac{1}{2}t^2,\ z = t^2 - 3t \qquad t: 0 \to 1$



$$\frac{dx}{dt} = 2t+3$$

$$\frac{dy}{dt} = t$$

$$\frac{dz}{dt} = 2t-3$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = (2t+3)^2 + t^2 + (2t-3)^2$$

$$= 4t^2 - 6t + 9 + t^2 + 4t^2 - 6t + 9$$

$$= 9t^2 + 18$$

so $\quad S = \int_0^1 3\sqrt{t^2+2}\, dt = \frac{3\sqrt{3}}{2} + 3 \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$