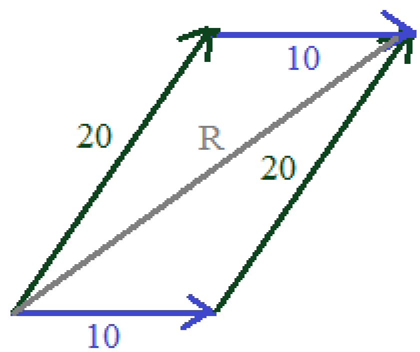
VECTORS

Notes, examples, and practice exercises (w/solutions)

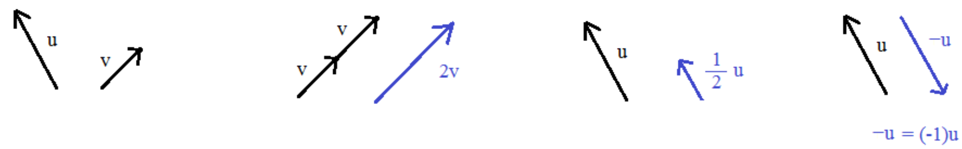


Includes matrices, unit vector, resultant vectors, law of cosines, dot product, navigation, and more!

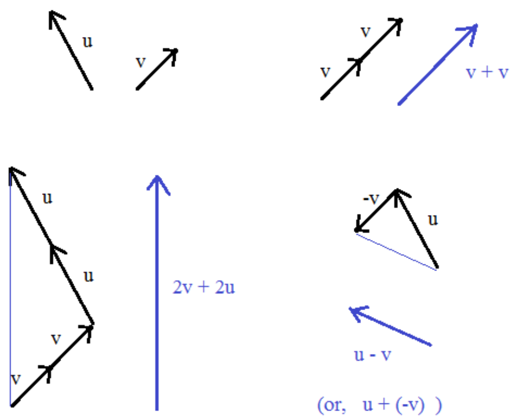
Vector Notes and Review

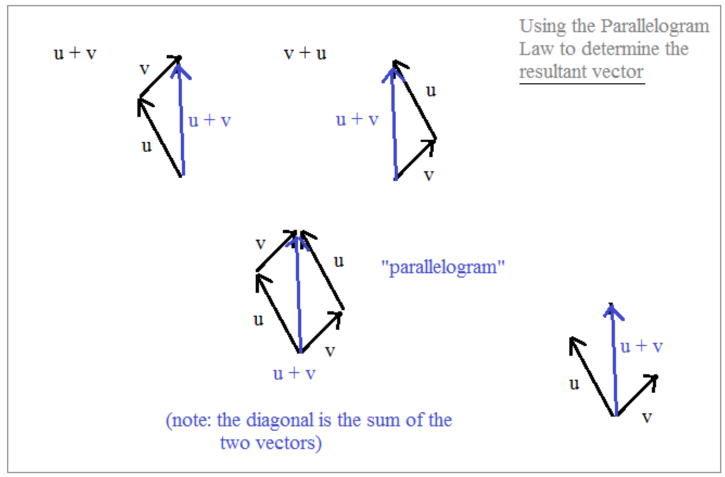
Scalar -- A quantity with magnitude (but not direction) ; Examples may include mass, numbers, length, or elements.

You can increase or decrease the magnitude of a vector by multiplying by a scalar



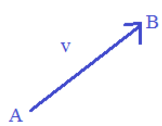
Vector Addition: "Tail to Tip"



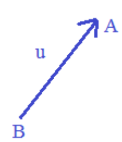


Vector Symbol $\rightarrow$

$v = \overrightarrow{AB}$



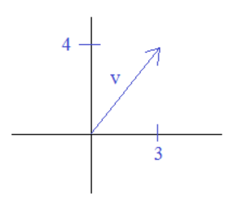
$u = \overrightarrow{BA}$



The vector arrow symbol describes direction from endpoint to endpoint.

Vector notation:

Examples: $v = 3i + 4j$ $\quad$ $v = \langle 3, 4 \rangle$ $\quad$ $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\quad$ $v = (3, 4)$



The notation contains the components of the vector.

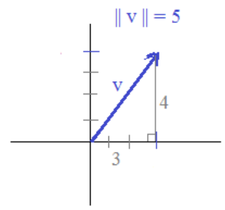
Magnitude: If $r = (x, y)$ represents vector displacement (from the origin), then the magnitude of r is

$$\| r \| = \sqrt{x^2 + y^2}$$

Note: the absolute value symbol may be used to indicate magnitude of a vector. $| r | = \sqrt{x^2 + y^2}$

Example: $v = (3, 4)$

the magnitude of vector v is $\sqrt{3^2 + 4^2} = 5$

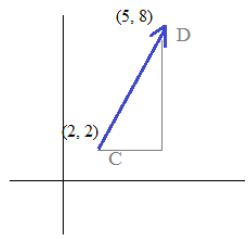


Pythagorean Theorem Confirms Vector Magnitude

Example: vector $v = \overrightarrow{CD}$ where C(2, 2) and D(5, 8) are the coordinates

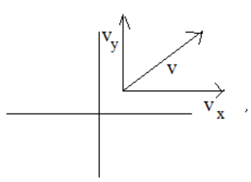
The magnitude is simply the length (or distance) of $\overline{CD}$.

$$\sqrt{(2 - 5)^2 + (2 - 8)^2} = 3\sqrt{5}$$



Component Vectors: Vectors parallel to specified (usually perpendicular) axes, whose sum equals a given vector.

Example:



The component vectors of V are V_x and V_y

$V_x + V_y = V$

(The 'resultant vector' of V_x and V_y is V)

Unit Vector and *Normalized Vector:* A unit vector has a magnitude of 1.
A vector can be normalized -- changed to a unit vector that is parallel (i.e. same direction)

$\hat{u}$ is the unit vector

u is the vector

|| u || is the magnitude

$$\hat{u} = \frac{u}{\| u \|}$$

note: caret symbol is often used to indicate a normalized vector.

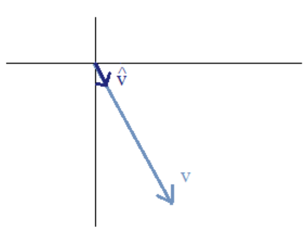
> If v is a unit vector, then $\|v\| = 1$

Example: Find the unit vector of $v = (4, -7)$

$\|v\| = \sqrt{65}$ $\quad \hat{v} = \left(\frac{4}{\sqrt{65}} \; \frac{-7}{\sqrt{65}} \right)$

A quick check: the magnitude of $\hat{v}$ is 1

$$\| v \| = \sqrt{\left(\frac{4}{\sqrt{65}}\right)^2 + \left(\frac{-7}{\sqrt{65}}\right)^2} = \sqrt{\frac{16}{65} + \frac{49}{65}} = 1$$

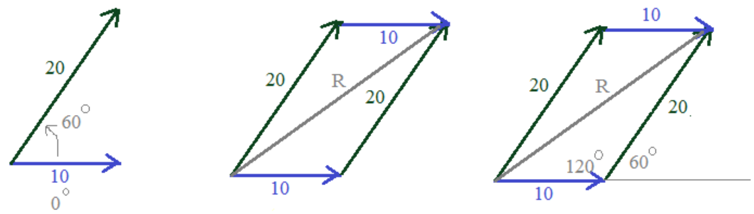


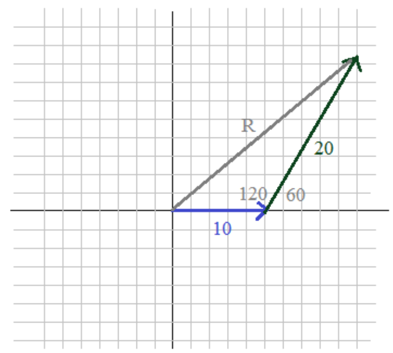
"Resultant Vector"

Example: Vector u has a magnitude of 10 and a direction of 0 degrees
Vector v has a magnitude of 20 and a direction of 60 degrees

Find the magnitude and direction of the resultant vector.

Step 1: Sketch and Use Geometry

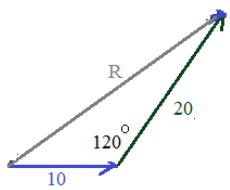




Step 2: Extract triangle and use Trigonometry

Magnitude of Resultant (length of R)

Using Law of Cosines:
$c^2 = a^2 + b^2 - 2ab(\cos C)$

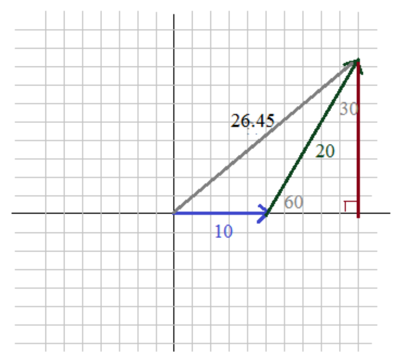


$R^2 = 10^2 + 20^2 - 2(10)(20)\cos 120°$

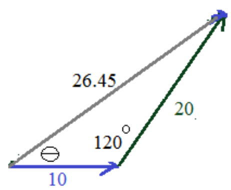
$R^2 = 500 + 400(-1/2)$

$R = \sqrt{700} \cong$ **26.45**

Direction of the Resultant (angle from horizontal 0)



Using Law of Sines:
$\frac{\text{SineA}}{a} = \frac{\text{SineB}}{b} = \frac{\text{SineC}}{c}$



$\frac{\sin(120)}{26.45} = \frac{\sin(\theta)}{20}$

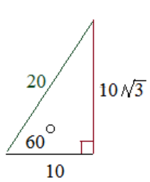
$\frac{.866}{26.45} = \frac{\sin(\theta)}{20}$

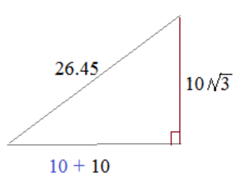
$\sin(\theta) = .6548$ **$\theta = 40.9°$**

Step 3: Check your work

Observe the graphs on the right. Using basic trigonometry values and the pythagorean theorem, you can confirm the values!

Vector v
direction:
60 degrees
magnitude:
20





Magnitude:

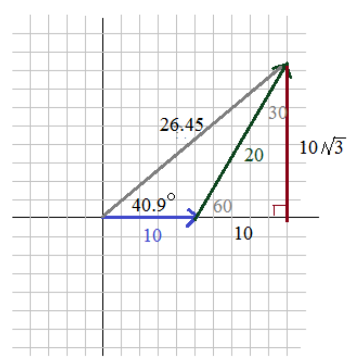
$20^2 + 10\sqrt{3}^2 = 26.45^2$

$400 + 300 = 699.6$ ✓

Direction:

$\text{Tan}(40.9) = \frac{10\sqrt{3}}{(10 + 10)}$ ✓

$.866 = .866$



Dot Product: Given two vectors $u = \langle x_1, y_1 \rangle$ and $v = \langle x_2, y_2 \rangle$

the dot product of u and v is $u \cdot v = x_1 x_2 + y_1 y_2$

Note: The dot product may be called the scalar product

Examples:

$u = \langle 2, 4 \rangle$ $v = \langle -3, 1 \rangle$

$u \cdot v = (2 \times -3) + (4 \times 1) = -2$

$a = 3i + 6j$ $b = i + 5j$

$a \cdot b = (3 \times 1) + (6 \times 5) = 33$

$u = (1, 1)$ $v = (3, -3)$

$u \cdot v = (1 \times 3) + (1 \times -3) = 0$

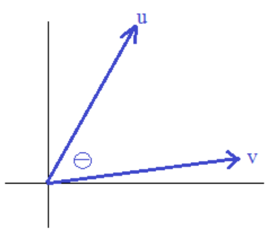
Note: If the dot product of 2 vectors is 0, then the vectors are *orthogonal* -- i.e. lie at right angles; are perpendicular

The dot product helps determine the angle between two vectors:

$$\cos \theta = \frac{a \cdot b}{\|a\| \, \|b\|}$$

where θ is the angle between vectors a and b.

Examples: Find the angle between $u = (4, 7)$ and $v = (6, 1)$

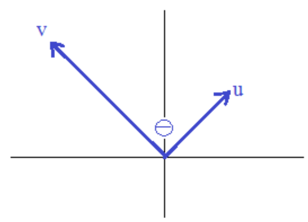


$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$$\cos \theta = \frac{31}{\sqrt{65}\,\sqrt{37}} = .6321$$

$$\theta = 50.8^\circ$$

Find the angle between $u = \langle 3, 3 \rangle$ and $v = \langle -6, 6 \rangle$



$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

$$\cos \theta = \frac{0}{\sqrt{18}\,\sqrt{72}}$$

$$\theta = 90^\circ$$

3-Dimensional Vectors: Extend the same formulas

Examples:

$A = (3, 5, 2)$ $B = (-1, 2, 1)$

Vector Addition $A + B = (2, 7, 3)$

Magnitude $\|A\| = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38}$ $\|B\| = \sqrt{-1^2 + 2^2 + 1^2} = \sqrt{6}$

Dot Product $A \cdot B = (3 \times -1) + (5 \times 2) + (2 \times 1) = 9$

Angle between vectors

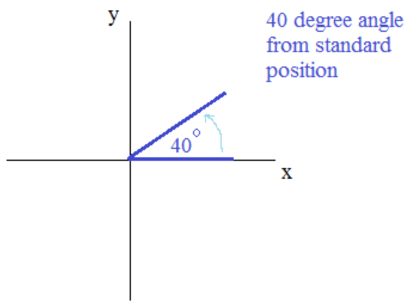
$$\cos \theta = \frac{a \cdot b}{\|a\| \, \|b\|}$$

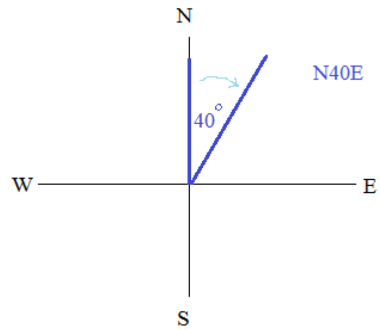
$$\cos \theta = \frac{9}{\sqrt{38}\,\sqrt{6}} = .596$$

$$\theta = 53.4^\circ$$

"Navigation vs. Graphing"

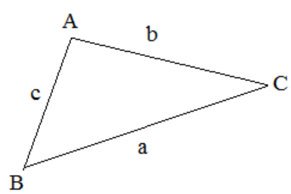
When graphing on a Cartesian Plane (x,y coordinate plane), the initial position is 0 and angles go *counterclockwise*. But, when using navigation, 0 may start at 'North' and angles go *clockwise*.





Trigonometry Review:

Law of Cosines -- When you know the lengths of 2 sides and the measure of the <u>included</u> angle, other parts of a triangle can be determined.

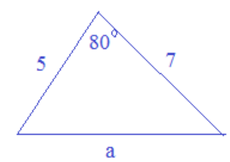


$$a^2 = b^2 + c^2 - 2bc(\cos A)$$

$$b^2 = a^2 + c^2 - 2ac(\cos B)$$

$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

Example:

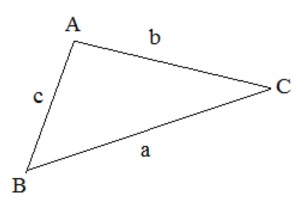


$$a^2 = (7)^2 + (5)^2 - 2(7)(5)\cos 80^\circ$$

$$= 49 + 25 - 70(.1736) = 61.85$$

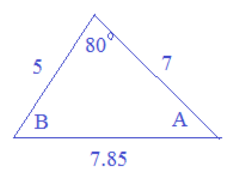
$$a \cong 7.85$$

Law of Sines -- Relation between interior angles of a triangle and their opposite sides are as follows:



$$\frac{\text{Sine}A}{a} = \frac{\text{Sine}B}{b} = \frac{\text{Sine}C}{c}$$

Example:



$$\frac{\sin 80^\circ}{7.85} = \frac{\sin B}{7} \qquad \frac{\sin 80^\circ}{7.85} = \frac{\sin A}{5}$$

$$\text{Sin}B = \frac{7(.985)}{7.85} \qquad \text{Sin}A = \frac{5(.985)}{7.85}$$

$$B \cong 61.4^\circ \qquad A \cong 38.9^\circ$$

Geometry Review:

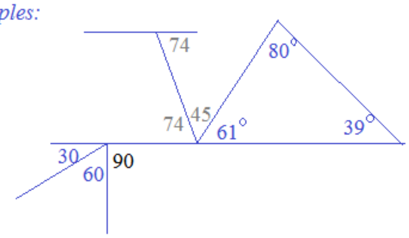
parallel lines cut by a transversal ---> alternate interior angles are congruent

sum of interior angles of a triangle ---> 180 degrees

sum of angles in a straight angle ---> 180 degrees

sum of angles in a right angle ---> 90 degrees

Examples:



Navigation Example:

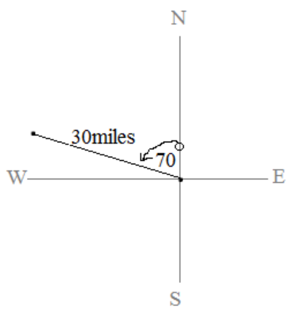
A math explorer leaves his home base and travels in the direction N 70°W. He travels 30 miles and reaches the rest station. The next week, he travels 50 miles in the direction N 10°E, reaching his destination.

a) Find the distance between the home base and the destination.

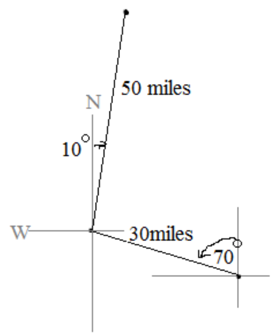
b) Find the bearing from the final destination back to the home base.

Step 1: Draw a picture

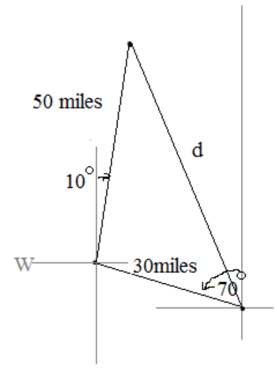
(Home to Rest station)



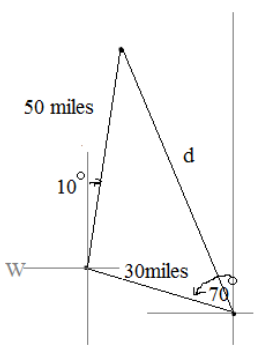
(Rest Station to Destination)



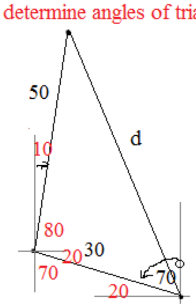
(Destination to Home)

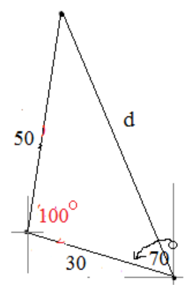


Step 2: Extract the triangle and find distance d



Use geometry relations to determine angles of triangle





Law of Cosines to find d

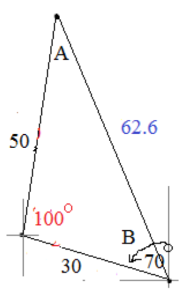
$a^2 = b^2 + c^2 - 2bc(\cos A)$

$d^2 = (30)^2 + (50)^2 - 2(50)(30)\cos 100^\circ$

$d^2 = 900 + 2500 - 3000(-.174) = 3922$

$d \cong 62.6$ miles

Step 3: Fill in triangle with angle measurements and find bearing



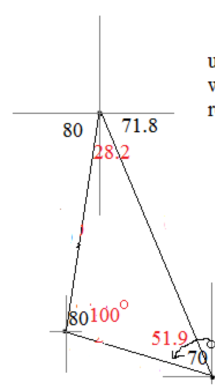
Law of Sines to find A and B:

$\frac{\sin 100^\circ}{62.6} = \frac{\sin A}{30}$ $\qquad$ $\frac{\sin 100^\circ}{62.6} = \frac{\sin B}{50}$

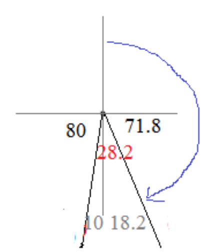
$\frac{.985}{62.6} = \frac{\sin A}{30}$ $\qquad$ $\frac{.985}{62.6} = \frac{\sin B}{50}$

$\sin A = .472$ $\qquad$ $\sin B = .787$

$A \cong 28.2^\circ$ $\qquad$ $B \cong 51.9^\circ$

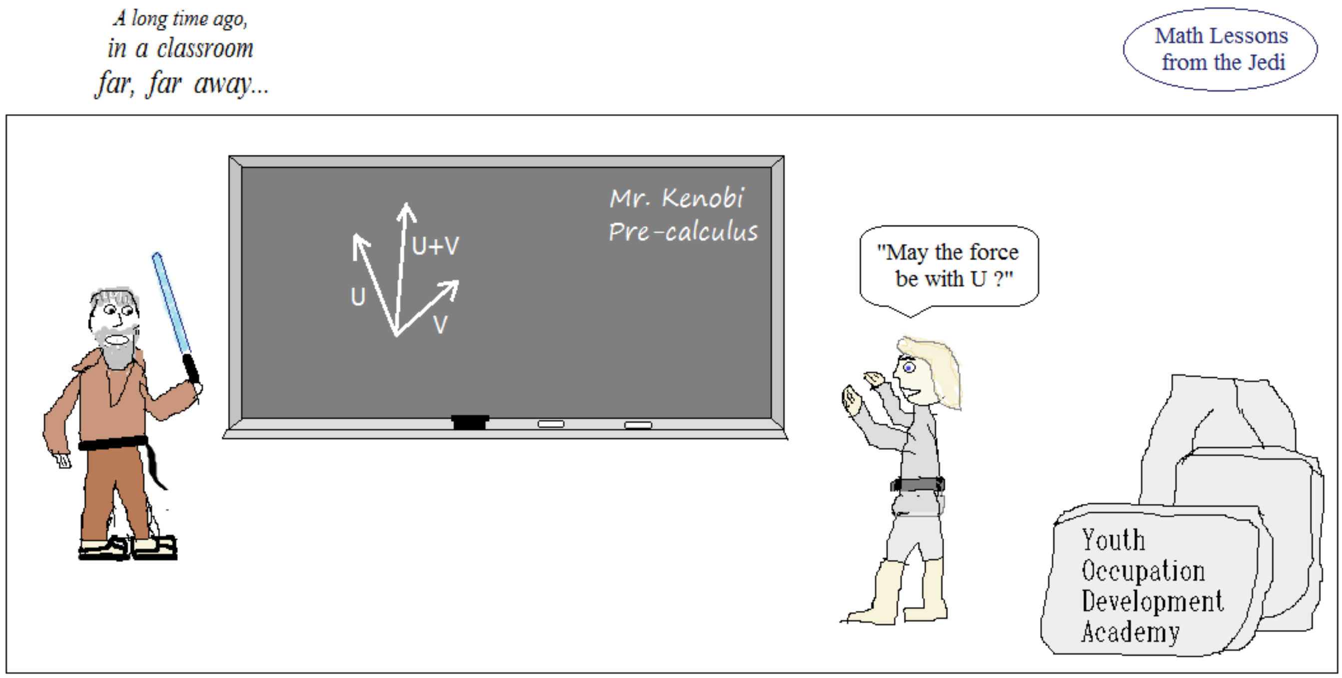


using geometry rules, we can determine relevant angles



The bearing is N 161.8° E
or S 18.2° E
or N 198.2° W

Note: Using horizontal and vertical axes maintain consistent bearings and help determine angle measurements.



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Obi-Wan teaches Luke about resultant vectors and (the) force

Introduction to Vectors Test (and Solutions)

Introduction to Vectors Test

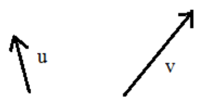
I. Vector Operations

u = 2i + 3j   v = i − 4j

a) 2u   b) u − v   c) v − u   d) ||u||

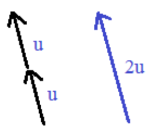
II. Sketching

Given:



Sketch: a) 2v   b) -3u

Example: 2u



c) u + v   d) u − v

III. Word Problems

1) A hiker leaves his car and goes 7 miles due North. Then, he travels 5 miles West, 4 miles South, and 4 miles East. How far is he from his car?

2) A plane is flying due east at an *air* speed 450 miles per hour. There is a southeast tailwind of 50 miles per hour.

a) Draw a diagram that represents the *ground* speed and direction of the plane.

b) Determine the ground speed and direction of the plane.

IV. More vector operations

1) $u = < 3, -2 >$ $v = < 2, 1 >$

a) what is $u + v$?

b) find the magnitude of v

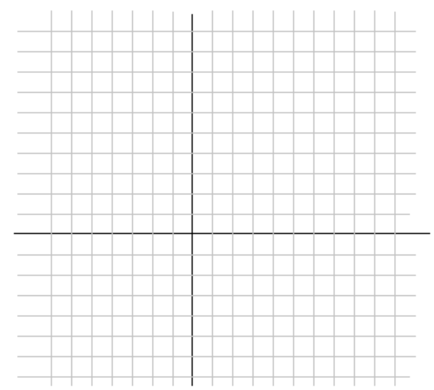
c) what is the 'normalized' vector of u? (i.e. write the unit vector of u in terms of i and j)

d) $u \bullet v =$

2) The endpoints of vector $\overrightarrow{AB}$ are A (2, -1) and B (3, 5)

a) Graph the vector $\overrightarrow{AB}$

b) Find and graph the standard vector (or, 'component vector') $\overrightarrow{OP}$ where $\overrightarrow{OP} = \overrightarrow{AB}$



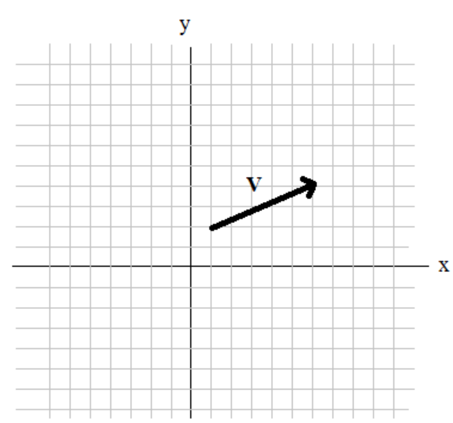
3) From the graph, determine the following:

a) The x-component $V_x =$

b) The y-component $V_y =$

c) magnitude $|V| =$

d) direction of vector V



V: Three-Dimensional Vectors

1) Find the angle between the following (3-dimensional) vectors: $u = 5i - 3j$

$v = -2j + k$

2) Find the vector with the same *direction* as $< 2, -5, -8 >$

and the same *magnitude* as $< -5, 1, 3 >$

Introduction to Vectors Test

SOLUTIONS

I. Vector Operations

$u = 2i + 3j \qquad v = i - 4j$

a) $2u$

$2(2i + 3j) =$

$4i + 6j$

b) $u - v$

$2i + 3j - (i - 4j) =$

$i + 7j$

c) $v - u$

$i - 4j - (2i + 3j) =$

$-i - 7j$

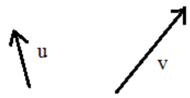
d) $\|u\|$

$\sqrt{2^2 + 3^2} =$

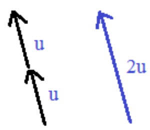
$\sqrt{13}$

II. Sketching

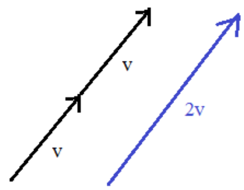
Given:



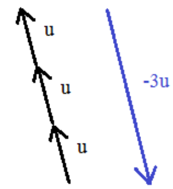
Example: 2u



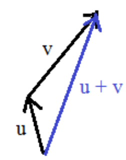
Sketch: a) 2v



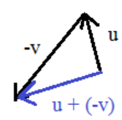
b) -3u



c) u + v

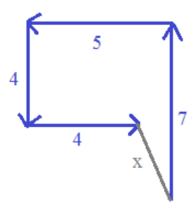


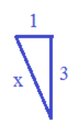
d) u − v



III. Word Problems

1) A hiker leaves his car and goes 7 miles due North. Then, he travels 5 miles West, 4 miles South, and 4 miles East. How far is he from his car?





(pythagorean theorem)

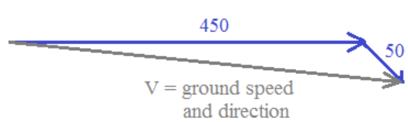
$\sqrt{1^2 + 3^2} = \sqrt{10}$ miles

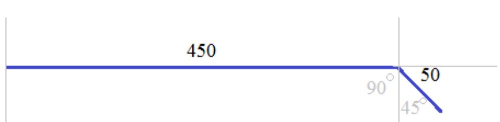
2) A plane is flying due east at an *air* speed 450 miles per hour. There is a southeast tailwind of 50 miles per hour.

a) Draw a diagram that represents the *ground* speed and direction of the plane.

b) Determine the ground speed and direction of the plane.

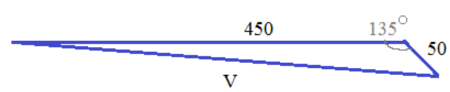
a)





ground speed: 487 mph
direction: N94.15°E

b)



Use Law of cosines to find V:

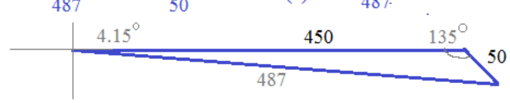
$V^2 = 50^2 + 450^2 - 2(50)(450)\cos(135°)$

$= 2500 + 202500 - 45000(-.707) = 236815$

$V = 487$ miles per hour

Use Law of sines to find angle:

$\frac{\sin(135)}{487} = \frac{\sin(x)}{50} \qquad \sin(x) = \frac{50(.707)}{487} = .0726 \qquad x = 4.15°$



IV. More vector operations

SOLUTIONS

1) $u = \langle 3, -2 \rangle$ $v = \langle 2, 1 \rangle$

a) what is $u + v$?

$\langle 3 + 2,\ -2 + 1 \rangle = \langle 5, -1 \rangle$

b) find the magnitude of v

$\|v\|$ or $|v| = \sqrt{2^2 + 1^2} = \sqrt{5}$

c) what is the 'normalized' vector of u? (i.e. write the unit vector of u in terms of i and j)

$u = \langle 3, -2 \rangle$ ----> $3i - 2j$

$\|u\| = \sqrt{13}$

$\hat{u} = \frac{3}{\sqrt{13}} i - \frac{2}{\sqrt{13}} j$

$\langle \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \rangle$

d) $u \cdot v =$

$(3 \times 2) + (-2 \times 1) = 4$

2) The endpoints of vector $\overrightarrow{AB}$ are A (2, -1) and B (3, 5)

a) Graph the vector $\overrightarrow{AB}$

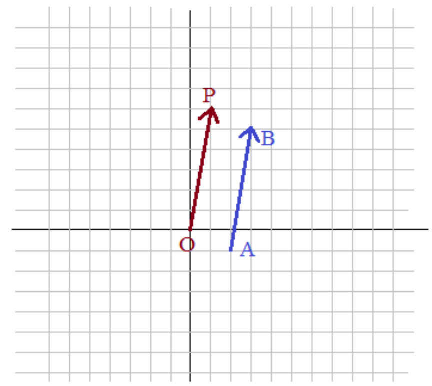
b) Find and graph the standard vector (or, 'component vector') $\overrightarrow{OP}$

where $\overrightarrow{OP} = \overrightarrow{AB}$

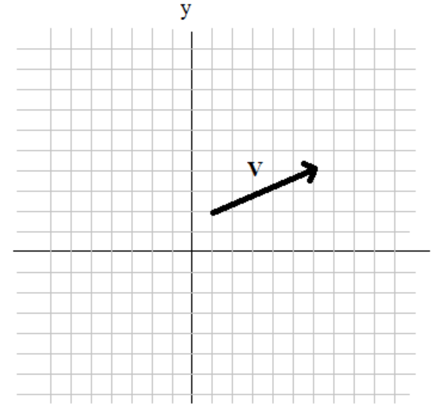
O (0, 0) origin

P (1, 6) $x_2 - x_1 = 3 - 2 = 1$

$y_2 - y_1 = 5 - (-1) = 6$



3) From the graph, determine the following:

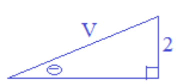


a) The x-component V_x = 5 (units)

b) The y-component V_y = 2 (units)

c) magnitude $|V| = \sqrt{5^2 + 2^2} = \sqrt{29}$

d) direction of vector V



direction is 21.8 degrees up from horizontal axis

$\tan(\theta) = \frac{2}{5} = .40$

$\theta = 21.8°$

V: Three-Dimensional Vectors

Solutions

1) Find the angle between the following (3-dimensional) vectors: $u = 5i - 3j$

$v = -2j + k$

$u = 5i - 3j + 0k$

$v = 0i - 2j + k$

$$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}$$

$$u \cdot v = (5 \cdot 0) + (-3 \cdot -2) + (0 \cdot 1) = 6$$

$$\|u\| = \sqrt{(5)^2 + (-3)^2 + (0)^2} = \sqrt{34}$$

$$\|v\| = \sqrt{(0)^2 + (-2)^2 + (1)^2} = \sqrt{5}$$

$$\cos\theta = \frac{6}{\sqrt{170}}$$

$$\theta = 62.6^\circ$$

2) Find the vector with the same *direction* as $< 2, -5, -8 >$

and the same *magnitude* as $< -5, 1, 3 >$

Step 1: find unit vector of $< 2, -5, -8 >$

$$\|v\| = \sqrt{(2)^2 + (-5)^2 + (-8)^2} = \sqrt{93}$$

$$\left< \frac{2}{\sqrt{93}}, \frac{-5}{\sqrt{93}}, \frac{-8}{\sqrt{93}} \right>$$

Step 2: find the magnitude of the 2nd vector

$$\|w\| = \sqrt{(-5)^2 + (1)^2 + (3)^2} = \sqrt{35}$$

Step 3: multiply the magnitude of the 2nd vector by the unit vector;
this gives you the correct direction and length!

$$\left< \frac{2\sqrt{35}}{\sqrt{93}}, \frac{-5\sqrt{35}}{\sqrt{93}}, \frac{-8\sqrt{35}}{\sqrt{93}} \right>$$

Let $A = \begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} x \\ y \end{bmatrix}$ where x and y are non-zero

Find any possible values of the *scalar constant* k, where

$$AB = kB$$

$AB = \begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x + 6y \\ 2x - y \end{bmatrix}$ $\quad kB = k\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$

If $AB = kB$, then

$\begin{bmatrix} -2x + 6y \\ 2x - y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$ $\quad kx = -2x + 6y \quad k = \dfrac{-2x + 6y}{x}$

$ky = 2x - y \quad k = \dfrac{2x - y}{y}$

substitution/set equations equal to each other

$$\frac{-2x + 6y}{x} = \frac{2x - y}{y}$$

cross multiply

$$6y^2 - 2xy = 2x^2 - xy$$

algebra/factoring

$$2x^2 - xy + 2xy - 6y^2 = 0$$

$$(2x - 3y)(x + 2y) = 0$$

Suppose x = 3; then, y = 2 can satisfy the equation

(3, 2) $\quad (2(3) - 3(2))((3) + 2(2)) = 0 \times 7 = 0$

$AB = \begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$kB = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ so, k = 2 because

$2B = 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Suppose x = 4; then, y = -2 can satisfy the equation

(4, -2) $\quad (2(4) - 3(-2))((4) + 2(-2)) = 14 \times 0 = 0$

$AB = \begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \end{bmatrix}$ so, k = -5 because

$-5B = -5\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} -20 \\ 10 \end{bmatrix}$

Possible values of k: -5, 2

Try x = 1; then, y = -1/2 can satisfy the above equation...

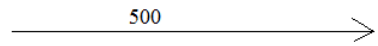
$AB = \begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5/2 \end{bmatrix}$ Again, k = -5 because $-5\begin{bmatrix} 1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5/2 \end{bmatrix}$

Vectors & Law of Sines/Cosines: Applications

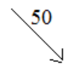
Example: An airplane flies due East at an air speed of 500 miles per hour.
A crosswind flows (from the Northwest) toward the Southeast at a rate of 50 miles per hour.

What is the *ground speed* and direction of the airplane?

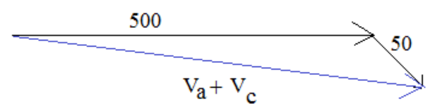
Airplane can be expressed as a vector: V_a



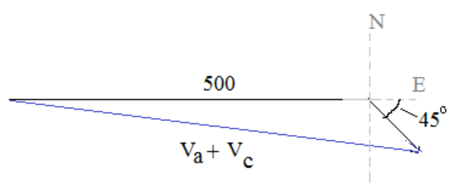
Crosswind can be expressed as a vector: V_c

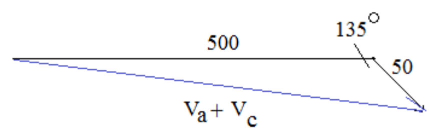


The groundspeed is the sum of the vectors...



We can transform the vectors into a triangle:





Use Law of Cosines to find ground speed of airplane:

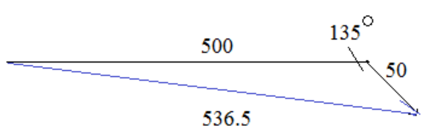
$$c^2 = a^2 + b^2 - 2ab(\cos C)$$

$$= (500)^2 + (50)^2 - 2(500)(50)(\cos 135)$$

$$= 250000 + 2500 - 50000(.707)$$

$$= 287{,}855$$

$$c \cong 536.5 \text{ miles}$$



Using Vectors:

$$V_a = 500i + 0j$$

$$V_c = \frac{50}{\sqrt{2}}i - \frac{50}{\sqrt{2}}j = 25\sqrt{2}\ i - 25\sqrt{2}\,j$$

$$V_a + V_c = 535.35i - 35.35j$$

$$\text{groundspeed} = \|V_a + V_c\| = \sqrt{535.35^2 + (-35.35)^2}$$

$$= 536.5$$

$$\text{direction} = \arctan[(-35.35)/535.35] = -3.8^\circ$$

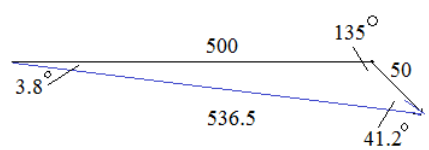
Then, use the Law of Sines to find the direction:

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin(135)}{536.5} = \frac{\sin(B)}{50}$$

$$\sin(B) = \frac{50\sin(135)}{536.5}$$

$$B = 3.8^\circ$$



The plane is going N93.8E
or
S86.2E
or
3.8 degrees south of due east

Thanks for visiting. (Hope this helped!)

If you have questions, suggestions, or requests, let us know.

Good luck!