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Research Article

On the Stone-Weierstrass theorem as a vital result in the study of the algebra of continuous functions on a Compact Hausdorff space

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Abstract

In this paper, we present the different versions and formulations of the Stone- Weierstrass theorem that makes it a vital result in the study of the algebra of continuous functions on a compact Hausdorff space. Instead of the real interval [a,b], an arbitrary compact Hausdorff space X is considered and instead of the algebra of polynomial functions, approximation with elements from more general sub-algebras of C(X) is considered. Some of its contributions and impact to the study of the algebra of continuous functions are also highlighted.

Keywords: Compact space; Hausdorff space; Locally compact; Algebra; Sub-algebra; Seperability of polynomials.

Introduction

The original version of Stone-Weierstrass theorem was established by Karl Weierstrass in 1885 using the Weierstrass transform [1]. Marshall H. Stone considerably generalized the theorem [2] and simplified the proof as seen in [3]. His result is known as the Stone-Weierstrass theorem. The Stone-Weierstrass theorem generalizes the Weierstrass approximation into two directions.

Stone starts with an arbitrary compact Hausdorff space X and considers the algebra C(X,$\mathbb{R}$) of real-valued continuous functions on X, with the topology of uniform convergence. He wanted to find sub-algebras of C(X,$\mathbb{R}$) which are dense. It turns out that the crucial property that a sub- algebra must satisfy is that it separates points. A set A of functions defined on X is said to separate points if, for every two different points x and y in X there exists a function P in A with the property that P(x)≠P(y).

Research methodology

Definition1.0 [1, Definition 2.4]

A space X is said to be compact if every open covering $\dot{A}$ of X contains a finite sub collection that also covers X.

Definition 1.1 [2, Definition 1.15]

A topological space X is called a Hausdorff space if for each pair x_1, x_2 of distinct points of X,there exist neighbourhoods $U_1 \, and \, U_2$ of $x_1 \, and \, x_2$ respectively that are disjoint.

Definition 1.2 [3, Definition 2.3]

A space X is said to be locally compact at x if there is some compact subset C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be locally compact. A compact space is automatically locally compact.

Definition 1.3 [4, Definition 3.6]

An algebra A over a field K is a vector space A over K such that each ordered pair of elements x,y in A, a unique product xy in A is defined with the properties

1) (xy)z = x(yz) for x,y,z in A

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2) x(y+z) = xy+xz, for x,y,z in A
3) (x+y) z = xz + yz for x,y,z in A
4) $\propto (xy) = x(\propto y)$, for x, y in A and scalars X

If k=$\mathbb{R}$ *or* $\mathbb{C}$ then A is real or complex.

Definition 1.4 [5, Definition 2.1]

A subset A of an algebra A is called a sub algebra of A if the application of the algebraic operations to elements of A yields again elements of A

Definition 1.5 [6, Definition 4.2]

Every polynomial has a splitting field that contains all its roots. There roots may all be distinct or there may be repeated roots. Let F be a field. A polynomial f(x) in F(x) of degree n is said to be separable if it has n distinct roots in some splitting field. Equivalently, f(x) is separable if it has no repeated roots in any splitting field.

Results and discussion

The Stone-Weierstrass approximation theorem [7]

Suppose f is a continuous complex-valued function defined on the real interval[a,b] $\forall \mathcal{E} > 0, \exists$ a P over C such that $\forall x \in [a,b],$ we have|f(x)-P(x)|$< \varepsilon$ or equivalently the supremum norm
$\parallel f - p \parallel \leq \varepsilon$
Remark 2.0.0: If f is real-valued, the polynomial function can be taken over $\mathbb{R}$

The statement of Stone-Weierstrass theorem

Suppose X is a compact Hausdorff space and A is a sub algebra of C(X,$\mathbb{R}$) which contains a non-zero constant function. Then A is dense in C(X,$\mathbb{R}$) iff it separates points.
Remark 2.0.1: This implies Weierstrass original statement since the polynomials on [a,b] form a sub algebra of $C_{[a,b]}$ which contains the constants and separates points.

Stone-Weierstrass theorem-real version [8]

The set $C_{[a,b]}$of continuous real-valued functions on [a,b] together with the supremum norm
$\parallel f \parallel = \underbrace{Sup}_{x\in[a,b]} |f(x)|$, is a Banach algebra (i.e an associative algebra and a Banach space such that
$\parallel fg \parallel \leq \parallel f \parallel \parallel g \parallel, for\ all\ f,g$

Remark

The set of all polynomial functions forms a sub-algebra of $C_{[a,b]}$(i.e a vector subspace of $C_{[a,b]}$ that is closed under multiplication of functions and the content of the Weierstrass approximation theorem is that this sub algebra is dense in $C_{[a,b]}$.

Stone-Weierstrass theorem (complex version)

Let X be a compact Hausdorff space and let S be a subset of C(X,$\mathbb{C}$) which separates points.Then the complex unital $*$-algebra generated by S is dense in C(X,$\mathbb{C}$).The complex unital $*$-algebra generated by S consitst of all those functions that can be obtained from the elements of S by throwing in the constant function 1 and adding them, multiplying them, conjugating them or multiplying them with complex scalars and repeating finitely many times.

Remark

This version implies the real version, because if a sequence of complex-valued functions uniformly approximates a given function f, then the real parts of those functions uniformly approximate the real part of f. As in the real case, an analog of this theorem is true for locally compact Hausdorff spaces.

Stone-Weierstrass theorem -Locally compact version [9]

Remark:

A version of the Stone-Weierstrass theorem is also true when X is only locally compact. Let $C_0(X,\mathbb{R})$ be the space of real-valued continuous functions on X which vanish at infinity; that is ,a continuous function f is in $C_0(X,\mathbb{R})$ if $\forall\ \mathcal{E} > 0\exists$ a compact set k$\subset X$ such that f<ε on X\K. Again, $C_0(X,\mathbb{R})$ is a Banach algebra with the supremum norm. A subalgebra A of $C_0(X,\mathbb{R})$ is said to vanish nowhere if not all of the elements of A simultaneously vanish at a point; that is $\forall x \in X, \exists f \in A{:}\, f(x) \neq 0.$ The theorem generalizes as follows:

Theorem

Suppose X is a locally compact Hausdorff space and A is a sub algebra of $C_0(X,\mathbb{R})$.Then A is dense in $C_0(X,\mathbb{R})$(given the topology of uniform convergence) iff it separates points and vanishes nowhere.

Remark

This version clearly implies the previous version in the case when X is compact, since in that case $C_0(X,\mathbb{R})= C(X,\mathbb{R})$

Lattice and Boolean Ring versions of Stone-Weierstrass theorem

Remark: Let X be a compact Hausdorff space.Stone's original proof of the theorem used the idea of Boolean rings inside $C(X,\mathbb{R})$; that is subsets B of $C(X,\mathbb{R})$ such that $\forall f,g \in B,$ the functions f+g and max{f,g} are also in B.

The Boolean ring version of the Stone-Weierstrass theorem

States that suppose X is a compact Hausdorff space and B is a family of functions in $C(X,\mathbb{R})$ such that

1. B separates points
2. B contains the constant function 1
3. If $f \in B$ then $af \in B\ \forall a \in \mathbb{R}$
4. B is a Boolean ring; that is if $f,g \in B$ then $f+g \in B$ and Max{f,g} $\in B$ then B is dense in $C(X,\mathbb{R})$

The lattice version of Stone-Weierstrass theorem [10]

Theorem 1

States that suppose X is a compact Hausdorff space with at least two points and L is a lattice in $C(X,\mathbb{R})$ with the property that for any two distinct elements x and y of X and any two real numbers a and b there exists an element f in L with f(x)=a and f(y)=b. Then L is dense in $C(X,\mathbb{R})$.

Theorem 2

More precisely lattice version can be stated as: Suppose X is a compact Hausdorff space with at least two points and L is a lattice in $C(X,\mathbb{R})$. The function φ in $C(X,\mathbb{R})$ belongs to the closure of L iff for each pair of distinct x and y in X and $\forall\varepsilon > 0, \exists$ some f in L for which |f(x)- $\varphi(x)| < \varepsilon$ and |f(y)- $\varphi(y)| < \varepsilon$.

Generalization of Stone-Weierstrass theorem (Bishops theorem) [5]

Another generalization of the Stone-Weierstrass theorem is due to the Errett Bishop.Bishop's theorem is as follows (Bishop 1961)

Theorem (Bishop's theorem) [5, Theorem 1.1]

Let A be a closed sub algebra of the Banach space C(X,ℂ) of continuous complex-valued functions on a compact Hausdorff space X. Suppose that $f|_s \in A_s \forall$ maximal set s$\subset X$ such that A_s contains no non-constant real functions. Then $f \in A$.

A New Version of Stone-Weierstrass Theorem for $(C(X),\|.\|)$ [12]

Due to the fact that the closure of a sub-algebra is a vector sub-lattice of C(X), therefore, the sufficient and necessary conditions for a vector sub-lattice V of C(X) to be dense in $(C(X),\|.\|)$ are also the sufficient and necessary conditions for a vector sub-algebra of C(X) to be dense in $(C(X),\|.\|)$ Let's have "The generalized Wierstrass approximation theorem"

Theorem. Stone-Weierstrass Theorem ("The generalized Wierstrass approximation theorem") [11, Theorem 1.3]

Let Z be a compact Hausdorff space. A vector sub-lattice or a sub-algebra V of C(Z)
is dense in $(C(X),\|.\|)$ if and only if
i) V separates points of Z, and
ii) for any f in C(Z), any x , y in Z, and any ε with 0 < ε < 1, there is a g in V such that
$|f(x) - g(x) < \varepsilon|$.and
$|f(y) - g(y) < \varepsilon|$.

Theorem. New Version of Stone-Weierstrass [12, Theorem 2.2]

Let Z be a compact Hausdorff space. A vector sub-lattice or sub-algebra V of C(Z) is
dense in $(C(X),\|.\|)$ if and only if
i) V separates points of Z, and
ii) for any x in Z, and any ε with 0 < ε < 1, there is a g in V such that $1- g(x)/\| g \| <\varepsilon$.

The contribution/ impact of Stone-Weierstrass theorem to the study of the algebra of continuous functions on a compact Hausdorff space

As a consequence of the Weierstrass approximation theorem, one can show that the space $C_{[a,b]}$ is separable: the polynomial functions are dense and each polynomial function can be uniformly approximated by one with rational coefficients; there are only countably many polynomials with rational coefficients.

The Stone-Weierstrass theorem can be used to prove the following statements which go beyond Weierstrass result:

i. If f is a continuous real-valued function defined on the set [a,b]x[c,d] and $\varepsilon > 0$, then $\exists$ a polynomial function P in two variables such that $|f(x,y)-P(x,y)| < \varepsilon \forall x \in [a,b] and\ y \in [c,d]$
ii. If X and Y are two compact Hausdorff spaces and $f: X \times Y \to \mathbb{R}$ is a continuous function then $\forall \varepsilon > 0, \exists\ n > 0$ and continuous functions $f_1, f_2, \dots f_n$ on X and continuous functions $g_1, g_2, \dots g_n$ on Y such that $\| f - \sum f_i g_i \| < \varepsilon$.
iii. The theorem has many other applications, including: Fourier series, the set of linear combinations of functions $e_n(x) = e^{2\Pi inx}, n \in \mathbb{Z}$ is dense in $C_{[0,1]\{0,1\}}$, where we identify the endpoints of the interval [0,1] to obtain a circle. An important consequence of this is that the e_n are an orthonormal basis of the space L^2 ([0,1]) of the square-integrable function [0,1].

Conclusions

The Stone-Weierstrass theorem is a vital result in the study of the algebra of continuous functions on a compact Hausdorff space. Because polynomials are among the simplest functions and because computers can directly evaluate polynomials, this theorem has both practical and theoretical relevance, especially in polynomial interpolation.

Conflicts of interest

Authors declare no conflict of interest.

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