Foundations For College Mathematics 2*e*

Prospectus

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Introduction:

Foundations for College Mathematics 2e is a remedial algebra textbook using a function approach with material based in real-world contexts. Guided discovery learning is implemented with an expectation of student understanding (as opposed to student memorization), and the use of the graphing calculator is required. At the same time, the table of contents of the traditional textbooks has been relatively preserved. It's just that the traditional topics are placed after a discovery study of function and function behaviors (Chapters Two & Three) – so that the concept of function and function behaviors can be used to teach these more traditional topics – more on this later.

Foundations was originally written for a developmental intermediate algebra course at the college level, and was published in preliminary versions in 1992 and 1994 by national-based publishers. Currently, it is self-published by Red Bank Publishing, first in 2000 and then with data up-dates in 2003, and now the second edition with a 2008 copyright.

About the Author:

Ed Laughbaum taught remedial mathematics (and many other courses) at Columbus State Community College for 24 years and then moved to The Ohio State University as Director of the Ohio Early College Mathematics Placement Testing Program and Associate Director of the Teachers Teaching with Technology College Short Course Program. Instead of teaching the 24 courses per year as he did at Columbus State, he now teach 3 – 4 courses per year, and the students are math teachers instead of undergraduate math students. Ed has given more that 200 talks at mathematics and mathematics teaching conferences throughout the world, and has had nearly forty professional works published in journals or commercial publications. He is actively involved in mathematics education as indicated by his role as editor of the *Ohio Journal of School Mathematics*, an editorial advisor for *Teaching Mathematics and its Applications*, as task force member for numerous Ohio DOE and Ohio Board of Regents projects, conference organizer, Director of the AMATYC Outer Banks Summer Institute, AMATYC standards "Crossroads-Revisited" section writer, and Associate Director of the Teachers Teaching with Technology College Short Course Program.

Target Audience:

At the school level, *Foundations* is typically used for a course in mathematics for juniors and/or seniors not prepared for pre-calculus after having difficulty in algebra I and/or II – yet still needing another math credit for graduation. *Foundations* is also used in a remedial algebra course where schools realize that offering a senior remedial course based in arithmetic is inappropriate given the times in which we live. For more information about the remedial course at the high school level, go to http://www.math.ohio-state.edu/~elaughba/ and click on the presentation "High School Mathematics Remediation through a Function Approach."

At the college level, *Foundations* is appropriate for intermediate algebra courses or as a beginning/intermediate algebra combined course. College developmental programs may choose to use any of the popular traditional texts for beginning algebra, but then use *Foundations* as the intermediate algebra text so that students are given a chance to really understand algebra and see its relevance to the real world – things that are often missing in traditional textbooks.

Rationale for the Concept of a Function Approach

Robin Rider, East Carolina University, on her dissertation research (Robin is not related to Red Bank Publishing and the email was unsolicited.): What I have done is compare two developmental math programs, one that teaches from a function approach, using multiple representations (MR) and graphics calculators. The other is a traditional university algebraic, no calculator approach. I gave them pre and post-tests. They could solve the problem using any method they wished. Hypothetically, both programs, which have the same concepts taught, should have been equally successful. However, what I have found is that not only were the students in the function, MR curriculum more successful (indicated by a higher percentage of correct problems on the post test), there were a large percentage who used more than one representation to solve the problem (or to check), thus their flexibility with the different representation was increased. (April 16, 2003 email)

Further:
From what I have analyzed so far, I think the qualitative interviews make it very clear that students' conceptual understanding of functions is significantly enhanced by teaching from a functional approach and utilizing and making connections between the multiple representations of functions. One of the critical pieces that I have found is that not only are the students more successful in the function curriculum but that they are even more successful when only using algebraic representations, thus not only were they more successful because they understood the table and the graph and could use them as an alternative tool, but algebraically their understanding was stronger. (October, 16, 2003 email)

From: Laughbaum, E. D. (2003). "Hand-held graphing technology in the developmental algebra curriculum". *MATHEMATICS AND COMPUTER EDUCATION, 34(1),* 63-71.

"The concept of function is one of the central ideas of pure and applied mathematics. For nearly a century, recommendations for school curricula have urged reorganization of school mathematics so that the study of functions is a central theme. Computers and graphing calculators now make it easy to produce tables and graphs for functions, to construct formulas for functions that model patterns in experimental data, and to perform algebraic operations on functions."

Chapter Five, page 42. Recommendations for High School Teacher Preparation. Conference Board of the Mathematical Sciences. 2001. *The Mathematical Education of Teachers*. American Mathematical Society. Washington D. C.

Susan E. Williams from the University of Houston says in her article "Effects of Hand-Held Computers on the Teaching of Algebra," <www.coe.uh.edu/insite/elec_pubHTML1997/ma_will.htm> (12/07/00) "The publication, 'Algebra for Everyone' (Edwards, 1990), suggested that all high school students need to study algebra. Several mathematics educators supporting this idea have noted that traditional algebra is approached in a manner that is too abstract for most students (Chambers, 1994; Hawkins, 1993; Heid, 1995; National Council of Teachers of Mathematics Board, 1993; Seeley, 1993; Silver, 1995; Steen, 1992; Usiskin, 1995). A recurring message in each of these articles is that traditional algebra often is taught using a rigorous approach that involves rote computations, meaningless manipulations of symbols, and acquisition of a predefined set of procedures for solving a fixed set of contrived problems." She continues to say, "One of the major themes that permeates mathematics in algebra and the courses that follow is the study of function. In spite of this, traditional algebra has been organized around the concept of equation and methods of solving equations. The concept of function has been 'patched in' at the end of the algebra course without providing substantial meaning or purpose."

It has long been the observation of the author that we rarely had much success in developmental algebra using a traditional curriculum and approach (Laughbaum, 1992). Generally speaking, students take remedial courses not knowing many of the basic manipulation skills upon entering and when they leave our courses, they still do not know many of the basic manipulation skills. So, what is gained? Textbooks and faculty not wanting to change drive the curriculum. Textbooks almost always use an equation approach and emphasize symbol manipulations. Using a function approach is a viable option for making a difference in the developmental curriculum.

"Many teachers and researchers know that the presentation of algebra almost exclusively as the study of expressions and equations can pose serious obstacles in the process of effective and meaningful learning."

Kieran, C. (1992). "The Learning and Teaching of School Algebra." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Grouws, D. A. pp. 390-419. Macmillan. New York.

An Argument for Using a Function Approach When Teaching Remedial Algebra/Algebra

Abstract:
This article makes the argument that algebra should be taught through a function approach implemented with a graphing calculator so that we can enhance learning based on recent research results in the cognitive sciences, and at the same time, hold to a higher standard of mathematical understandings through an appropriate level of rigor.

The literature research shows that:

- We remember algebra longer and have better memory by using associations – made through function permeating the content. That is, students are more likely to remember the mathematics taught because we capitalize on associations made through using a function approach.
- Learning is made simpler, faster, and more understandable by using pattern building as a teaching tool. In a function approach, almost all of the pencil and paper activities, e-teaching activities, and class discussions use pattern building to reach a generalization about a concept or skill.
- Students cannot learn if they are not paying attention. The graphing calculator is used to draw attention to the mathematics through its basic functionalities **including,** various app software.
- Without visualizations, students do not understand or remember the mathematics as well. In the function approach visualizations are used first before any symbolic development. This greatly increases the likelihood that students will remember the mathematical concept being taught.
- Considerable brain processing takes place in the unconscious side of the brain, including a learning module. To make this processing possible for our purposes, the brain must be primed. The function implementation module and early learning activities prime the brain for all the algebra that follows.
- The enriched teaching/learning environment promotes correct memory of math learned. The wide variety of teaching activities facilitated by the function approach provides the enriched environment.
- Contextual situations (represented as functions) provide meaning to the algebra learned. Algebra taught without meaning creates memories without meaning that are quickly forgotten.

The author argues for using a modified traditional content, but approached through function, which will reorder the content and capitalize on function concepts to develop understanding, long-term memory, and skills. As the author explores cognitive processes of associations, pattern recognition, attention, visualizations, priming, meaning, and the enriched teaching environment, he provides an argument that both the function approach and graphing calculators are crucial to teaching and learning of algebra.

Preface

Currently, developmental/remedial algebra (and arithmetic) constitutes more than 50% of the math sections offered at two-year colleges, and around 20% at four-year colleges. With incoming new freshmen graduating in June, the numbers are often higher. So, repeating high school algebra in college is nearly a standard, and it has been for many years. This data suggests there is a problem with the mathematical understandings of entering college students. Experience tells us that remedial students do not remember much of high school algebra; they often remember incorrectly, they do not understand much of what they learned, and/or they did not really learn algebra in high school. We likely cannot blame any one thing for this failure, but it is probably a result of many.

We might also wonder what is happening to the remedial students in college. That is, if high school students fail to learn algebra with understanding, long-term memory, and/or correct long-term memory, what is happening to these remedial students as they navigate through developmental algebra programs in college? Are developmental math programs successful? Currently, 42% of students taking a course in basic algebra fail or drop out. The rate is 38% in intermediate algebra. Further, college graduation rates for remedial students are relatively low compared to non-remedial students. So, maybe developmental algebra programs are not as successful as we would like.

At a gathering of developmental algebra professors (2005 AMATYC Conference), the concerns expressed were as follows:

- Students use a graphing calculator in high school (expressed as a negative)
- Students are not prepared mathematically
- Students have no study skills
- Student baggage brought to college
- Student tutoring issues
- Student drop out rate.

In this occurrence, perceived issues were student based, and by associated implication, maybe mathematics education in high schools.

Further confirmation that students are the perceived problem is found in current topics in professional journals. The *Journal of Developmental Education* lists the following articles relative to developmental math in the last two years:

- “Increasing Attendance Using Email: Effects on Developmental Math Performance”
- “Mathematics Self-Efficacy of College Freshmen”
- “Delaying Developmental Mathematics: The Characteristics and Costs”

There is little discussion of teaching approach, or content. By its absence in professional circles, it is implied that the problem isn’t the typical equation-solving approach we take when teaching the content.

Student issues are a part of the problem. But we hear the same conference presentations over and over, as each new generation of teachers "knows" the solution is to have activities and games that motivate students? Each new generation knows all we need are new activities so students can learn to factor/solve equations/etc. Some think the solution is found in emphasis on acronyms so that students can memorize procedural algebra. Others know that the use of teacher-developed activities with the traditional textbooks as the solution. Major publishers marginalize the power of the graphing calculator as a teaching tool, and novice users have the short-sighted vision that the graphing calculator is a tool that merely does the math they teach. Experienced remedial algebra teachers cling to publisher-enabled step-by-step instruction of step-by-step symbol manipulation. We seem to be stuck with solutions based in another time and for a different set of students.

There are many "fixes" for the current problems we face in remedial algebra, but none seem to be systemic and seamless to teaching and learning. None are scaleable – contrary to what "add-on" publishers would have us believe. The solution must be a systemic and seamless process that reforms the curriculum, pedagogy, and teaching tools. We need a process that has a rhythm that is in sync with current research.

In this paper, the author investigates issues of pedagogy, approach, content, and tools, and propose that teaching algebra through a function approach is part of the solution, both in high school programs and in remedial algebra programs in college. The teaching flow is natural. Everything fits – content, pedagogy, approach, and tools. Everyone can teach it after some initial professional development. There are no annoying add-ons. The process is seamless, and at no time do we assume students are the problem.

What is a Function Approach?

Teaching algebra from a function approach is using function, function representation, and function behaviors to teach algebraic concepts and skills. Function notation is introduced at an appropriate time after other representations are utilized. Formal $f(x)$ function notation is delayed because it is not integral to the teaching/learning process during the initial stages. The function approach to teaching algebra does not mean moving the function chapter from near the end to near the beginning of a textbook.

The above mentioned definition of a function approach implies that function is an underlying theme throughout a course in algebra – not just studied as a chapter or as part of a "content" or "concept" list as might be found in standards documents. It also suggests that a "function implementation module" is needed before any traditional algebra is taught [this is Chapters One and Two in *Foundations*]. The module provides content that begins with contextual real-world numeric representations of functions and leads to students learning to move freely through representations with a graphing calculator. This is followed by an analysis of the geometric behaviors of functions integrated with studying parameter-behavior connections. The implementation module facilitates teaching of a slightly revised and re-ordered traditional curriculum that allows us to capitalize on the cognitive learning concepts of associations, pattern recognition, attention, visualizations, meaning, priming, and an enriched teaching environment. These ideas play an extremely important role in teaching and learning of algebra, and are naturally and <u>seamlessly</u> integrated into the mathematics and pedagogy through using a function approach implemented with a graphing calculator. A graphing calculator is required for all students at all times – both in the implementation module and throughout the algebra course.

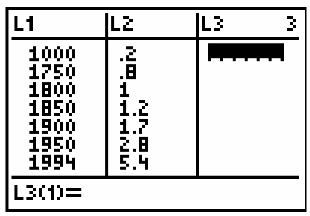
The <u>concept</u> of using a function approach to teaching/learning algebra is not new. Already in 1923 "[T]he National Committee on Mathematical Requirements, chaired by Ohio native J. W. Young, issues its report, *The Reorganization of Mathematics in Secondary Education*, recommending "functions" as a central concept in the high school mathematics curriculum." (Kullman, 27) In 2001 the Conference Board of the Mathematical Sciences released the book *The Mathematical Education of Teachers*, and in Chapter 5 it says "For nearly a century, recommendations for school curricula have urged reorganization of school mathematics so that the study of functions is a central theme. Computers and graphing calculators now make it easy to produce tables and graphs for functions, to construct formulas for functions that model patterns in experimental data, and to perform algebraic operations on functions." (42) While the authors of this document agreed that function should be a central theme, they did not specify methods for using function and function concepts, but provided a selection of functions to be included in the curriculum. The likelihood is that they had not seen a function-based algebra curriculum and did not understand how function behaviors and representation are used to complement basic brain learning processes. Further, when the document was written, graphing calculators just did as they describe "produce tables and graphs for functions … " They did not have the ability to execute software like the Transformation App, StudyCard App, and Cabri Jr, and they didn't have extremely easy-to-use data collection devices and probes. The apps extend the functionality of the graphing calculator and further support the cognitive features described in this paper. While CAS does exist on graphing calculators, it is rarely used to teach algebra in the US.

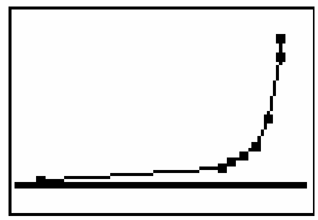
The Function Approach Implementation Module (Chapters Two and Three of *Foundations*)

The function approach process begins with a study of functions in numeric and graphic forms. The mathematics in the initial materials in the module is basic (when using a graphing calculator), classify given data relationships by shape, and whether/when they appear to be increasing and/or decreasing in nature. Initial materials in the implementation module (typically taught in two-three days) are used to analyze real-world relationships (data pairs with no traditional symbol manipulation or formal function notation). They integrate a variety of relationships such as linear, quadratic, exponential, absolute value, etc. simultaneously – just as students might encounter in their lives. For mathematical purposes, a variety of function types are needed so that students can categorize by type and recognize differences and similarities. From a learning perspective, we use functions presented in a contextual setting because "When a child has a personal stake in the task, he can reason about that issue at a higher level than other issues where there isn't the personal stake. These emotional stakes [real-world contexts that make sense to the students] enable us all to understand certain concepts more quickly." (Greenspan & Shanker, 241-2) We also start with the numeric and graphic representation to connect the new content with previous content.

Students learn to move freely from numeric to graphic forms and make the connections between the two using graphing technology. This is easily accomplished by making data sets available to students through calculator programs that are distributed to student devices via the GraphLink™ cable or through TI Navigator™. When the programs are executed, the data is transferred from the program to the list editor making it available to be viewed in numeric and graphic forms (and later in a more traditional symbolic form – see the section "The Function Approach Implementation Module Continued through Pattern Recognition"). A wide variety of function types can also be obtained through various data probes connected to the graphing calculator. The question of whether students think the relationships are increasing or decreasing can be answered by looking at the numeric or graphic representations, or it can also come from the data collection process. The concrete-physical activity provides the "emotional" connection, making learning simpler and faster. At the same time, students make the connection between increasing (or decreasing) numbers in the range with a rising (or falling) graph.

Example, world population:





The population of earth (shown above in billions and calendar years which are used to add meaning) is increasing as confirmed by the numeric representation, the graph, and the context. The shape of the graph might be called a "J" or backwards "L" by students – which the instructor can change to exponential when appropriate. But, students are asked to identify a shape, and for now, the letter "J" is fine. We see that the increasing behavior of people population implies a rising graphical representation.

This is just one example in the process. Students are given a wide variety of data sets representing various function types. – all from real-world contextual relationships to add the emotional connection. [all data sets are available as ancillary materials in *Foundations*]

Associations: How Students Remember What We Teach – A Temporary and Needed Diversion

> Note_1: Joseph LeDoux writes, "cognitive science deals with the way the mind typically works in most of us, rather that the way it works uniquely in any one of us." (24) Note_2: one would expect similarities between the recently referenced neuroscience/cognitive psychology research findings in this paper and learning theories based on educational research, because the brain/mind controls all behavior, thinking, and feeling.

LeDoux argues (as do most all cognitive scientists) that neurons firing together in a synaptic circuit cause associated circuits/patterns to fire. "The ability to form associations between stimuli is perhaps the benchmark test for synaptic mechanism of learning." (141) The point is that if one memory (something learned) is associated to another memory through shared neurons, synaptic circuits or memory module, activation of one memory will likely activate those associated. Hawkins makes the point that "… even though we have stored so many things, we can only remember a few at any time and can only do so in a sequence of associations." (73) So, during these first couple days of the implementation module, we have already made associations between algebra and real-world situations.

As you will see later, making associations is a "standard" in the function approach. Connecting algebra content through function allows us to create associations that will more likely help the brain recall what we teach. We know that our students are taught many things in many different ways, but in the algebra classroom we have a marvelous opportunity to structure our teaching and curriculum so that they can more likely recall what we have taught. Throughout the course we make associations among new material to be learned, previous content, and contexts through function and function representation. These associations are repeated on several occasions which increase synaptic strength that further increases the likelihood of creating a lasting memory of the algebra learned. The contexts are re-used on several occasions which also assists memory because, "In general, how well new information is stored in long-term memory depends very much on depth of processing, … A semantic level of processing, which is directed at the meaning [contexts] aspects of events, produces substantially better memory for events than a structural or surface level of processing." (Thompson & Madigan, 3)

As a simple example of how the neocortex uses associations to recall a memory (something learned), let's suppose you are trying to recall the name of a person upon seeing them out of their normal context. You can't remember. All you can come up with is something like fate, or the letter G. As soon as you think fate (or G) you think of gate which leads you to gates, and you blurt out Bill Gates. Hawkins makes the argument that our neocortex creates a memory as an electrical pattern of neurons firing. "Memory recall almost always follows a pathway of associations. One pattern evokes the next pattern, which evokes the next pattern, and so on." (71) In the case of algebra taught using a function approach, we create the ability of the brain to remember what we have taught by always associating new concepts or skills to one or more ideas previously taught and to real-world understandings. This is possible because some concept of function or function behavior is always applicable to the algebraic concept being taught. Further, we use contextual situations when developing a concept so that students can also associate algebra to real-world experiences, provide meaning, and provide cues so that students can remember the algebra longer. What we create in our students are memories that provide links to the mathematics taught. For example, how do we recall the first law of exponents? If you have made no associations to it, students are likely to have forgotten it after *x* weeks. But in the function approach, we have made many connections to the underlying concept.

- US debt, world population, TB bacteria, E. coli, Radon, depreciation, exponential function, behaviors of exponential functions, operations on exponential functions, exponential growth; other laws of exponents, related discovery activities, etc.

Schacter draws an interesting conclusion on making associations in his book *The Seven Sins of Memory: How the Mind Forgets and Remembers*. He says, "If associated details are bound together with an object or action, it becomes easier to recall …" (95) It is perhaps this idea that some teachers use when asking students to move arms, walk, stand up, etc. when memorizing something to be learned. Using the implied object or action as unrelated bodily movement may be a stretch. But what if it is germane to the lesson to be learned? Like, for example, when students learn the increasing/decreasing association to the graphical representation of time-distance as they walk and collect data – real-time. "Emotions [a meaningful context] help a child comprehend even what appear to be physical and mathematical relationships." (Greenspan & Shanker, 56)

A graphing calculator with all of its opportunity for novelty through motion, is an object, requires action in the learning process, and is associated to the mathematics. Schacter also makes the argument that if learning is rich with associated cues it is much easier to remember and is less affected by transience or blocking. (63) So it seems that we should associate the mathematics to be learned with an action or object, and add cues (contexts) to assist with recall. In the case of categorizing data sets by shape and identifying the increasing/decreasing behavior, we associate the mathematical concepts with a context, with previous concepts, with the action of data collection, and mix this with the cognitive advantages (see why later) of processing on a graphing calculator.

When a student takes the action of scrolling through the numeric representation of a data relationship like time-temperature, for example, it is often obvious whether the temperature is increasing or decreasing. That is, the context helps make the mathematics more understandable. We associate a cooling temperature with a decreasing relationship. Looking at the graphical representation, we now associate the mathematical concept of decreasing with that of a graph dropping. Schacter also observes "Any attempt to reduce transience [one kind of memory loss] should try to seize control of what happens in the early moments of memory formation, when encoding processes powerfully influence the fate of the new memory." (34) (Keep this thought in mind when we investigate the role of visualizations relative to memory – see the section "The Visual Brain.") So, as the student is learning a mathematical concept such as the increasing/decreasing behavior, we make the real-world associations (through contexts) <u>at the beginning of teaching the concept</u> – not at the end of the lesson as an application where the memory advantage is lost. Understandable real-world contexts provide an "emotional" connection to the mathematical concepts being taught. Greenspan & Shanker's research tells us that "When a child has a personal stake in the task, he can reason about that issue at a higher level than other issues where there isn't the personal stake. … These emotional stakes enable us all to understand certain concepts more quickly. … understanding concepts involves a sequence of steps that begins with emotional interactions." (241-2) In addition, they observe that "This double coding [emotional and mathematical] allows the child not only to "cross-reference" each experience and subsequent memory in mental "catalogues" of phenomena and feelings but also to reconstruct them when needed." (57)

The bottom line is that we need to enhance learning by teaching algebra using associations to increase the likelihood of the brain remembering what we teach. In teaching the traditional equation-solving approach to algebra, topics are often taught in isolation, usually with little connection from one to the other. This is often true even when algebra is approached through activities or "standards" based proposals. This implies that students are not as likely to remember what we teach. Historically, we know that teaching the traditional curriculum (or some reform materials) does not seem to cause long-term memory of algebra because of experiences with remedial students. One might ask if there is educational research to show that using a function approach does promote longer and more accurate memory. There may be little educational research on the subject to confirm or deny, but neuroscientific research is clear.

Getting Attention

"All the major theorists in the area of learning agree that information in a lesson cannot be learned if children are not paying attention." (Byrnes, 74) We also know that "… memory requires selective attention for encoding and for recall." (Kandel, 311) In the function approach the contexts of the data relationships are use to attract the attention of the student. We keep the attention by using handheld technology with its ability to provide novelty. You may not think of a graphing calculator as a tool for keeping attention – see more on this issue later. Langer argues for "… the importance of novelty to the process of paying attention." (49) So, to keep the attention of students, something must change (be novel). Examples might be scrolling through a table, seeing a graph displayed (or re-displayed after changing a parameter), changing screens during mathematical procedures, or seeing PowerPoint®-type presentations as the student moves through a teaching activity on something like the Texas Instruments StudyCard™ app. Attention and action are required by students as they drag a line around the screen while using Cabri Jr., for example, as they observe connections (making associations) between behaviors and function parameters. The motion and the changing parameters provide for the novelty.

"… [T]he literature has advanced enough to suggest that teachers can manage attention through the use of content that is interesting to students …" (Byrnes, 89) Experience tells us that students find the function approach to algebra more interesting than the traditional, and the functionality of the technology keeps the interest longer, as do using contextual situations by adding an emotional attachment. Maintaining student attentiveness on the mathematical objective can be difficult and requires activity much like you find on graphing calculators. "Very few teachers can effectively compete with the attention grabbing and holding power of computers." (http://darkwing.uoregon.edu/~moursund/Math/brain_science.htm) So we should not compete, but embrace the devices and adapt them to assist in the teaching/learning process.

Another tool used to address attention is questioning. When you ask your entire class a question, a student knows they may not have to answer. As such, they do not need to give you their attention, so no learning takes place in those students not paying attention. (Byrnes, 55, 74) Let's suppose that during class discussion you ask an individual student a question. In doing so, you demand the attention of that one student. But others in the class may choose to not be attentive to your teaching, so again you have lost the teaching moment to those students. The point is that asking a particular student a question gets their attention. So, what if your teaching lesson asked a series of questions of every student? Every student must reply; after which they receive a response from you, and then they are asked another question. And this process continues until the lesson is finished. Graphing technology provides the tool to maintain attention through StudyCard activities. Further, students are not inhibited by a negative peer stimulus (Mazur, 17) when responding to a question through technology, and are free to attend to the lesson. In addition, using the function approach allows us to make associations as discussed above embedded in the lesson, so we can also increase the likelihood that our students will remember the lesson on the StudyCard stack (for examples, see www.math.ohio-state.edu/~elaughba/).

Greenspan & Shanker's research (published in the book *The first idea: How symbols, language, and intelligence evolved from our primate ancestors to modern humans*) suggests that learning is facilitated through a co-regulated back-and-forth process between teacher and learner. StudyCard stack lessons can emulate this process. On the front of an "electronic" flash card, we ask a question. Students respond. On the back we include information and reasons for the correct answer. And then we ask another question. The process continues to the end of the lesson when students learn their score and are given a chance to answer incorrect questions again. Some teachers design StudyCard stacks to process memorizing of facts. The down side of this is that many (but not all) of the cognitive advantages are lost; the exception is that it may increase synaptic strength.

A much better way of increasing synaptic strength (which helps with long-term memory) is through distributed learning. That is, revisiting a concept many times – as is done in the function approach, will vastly improve long-term memory. The time-memory curve of a concept approaches a horizontal asymptote-like level that is much higher when taught through distributed learning. While "Bahrick and other researchers argue that findings such as these [distributed learning vs. "one shot lessons"] need to be taken seriously by administrators of training programs including the rather expensive one called education." (Thompson & Madigan, 92) The author would also argue that those teaching algebra need to implement teaching/learning strategies that are suggested by neuroscience and cognitive science research. This is especially true since it is relatively simple to do through a function approach and handheld technology. [distributed learning is a main feature of *Foundations*]

Hawkins argues that … "intelligence and understanding started as a memory system that fed predictions into the sensory stream. These predictions are the essence of understanding. To know something means that you can make predictions about it." (104) Teaching through questioning then, feeds into the constant and normal activity of the neocortex as it is searching for something to make a prediction about. And what if the questions asked through the StudyCard app are difficult and the student makes a wrong prediction about the correct answer? "When that prediction is wrong, your attention is immediately aroused." (Hawkins, 95) When we have the student's attention, – even when we ask a question that is difficult, we have the opportunity to teach through the back of each card which may contain explanations of the correct response, and why incorrect responses are incorrect. Again, the graphing calculator is a tool that can be used to deliver our lesson while capitalizing on cognitive processes.

The Function Approach Implementation Module Continues Through Pattern Recognition

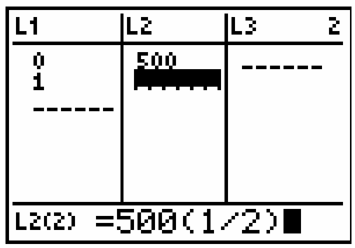
The next step in the implementation process raises the cognitive level by moving from the numeric and graphical representations of a function to the symbolic representation. This is not accomplished by students memorizing English-math conversions. Rather, we use pattern building and the list editor to reach the goal. Pattern recognition, followed by a generalization, is an innate function of the human brain, while English-math conversions are language specific and are more taxing on brain resources. Below are just a few examples of what research shows relative to pattern recognition:

- "Seeing the world in patterns increases understanding of how it works and leads to expectations and mastery, a scientific attitude." (Greenspan & Shanker, 64)
- "[T]he brain's capacity to generalize is astonishing. [T]he brains of higher-level animals autonomously construct patterned responses to environments that are full of novelty." (Edelman, 38-39)
- "We crave pattern … They reassure us that life is stable, orderly, and predictable." (Ackerman, 55-56)

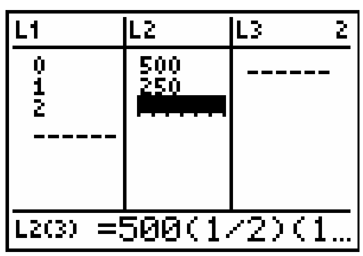
We want to capitalize on this natural occurring cognitive function as we teach most mathematical ideas, and as we use exercises (homework) to increase synaptic strength. Below is an example that uses pattern building to develop the symbolic representation of a function. It is one simple example of how pattern building is used in concert with guided discovery. Note: this piece of the implementation module (moving from numeric/visual to symbolic) is typically taught in two days when used with remedial algebra students. This process is not a difficult teaching task because of the power of pattern recognition and the technology – even though it is a major mathematical idea. "All your brain knows is patterns." (Hawkins, 56) So again, we find ourselves teaching to how the brain processes understandings.

Suppose we have 500 M & M candies (initial condition) and we toss them on the table. How many do we expect to have the M facing up?

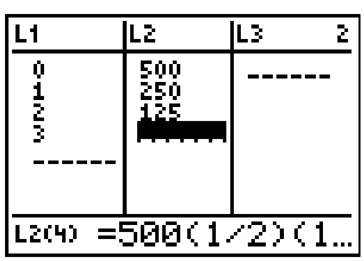
Students typically say 250. You respond, "How did you get that, and why does your answer make sense?" (see the edit line for the student answer)



We eat the M & M's with the M facing up (to help activate the emotional attachment), and toss the remaining M & M's on the table. We now have about 250 M & M candies on the table. How many do we expect to have the M facing up?
Students typically say: 125: You respond, "How did you get that?"
Students may say: 250(1/2): You say, "And where did the 250 come from?" (see edit line for the student answer)

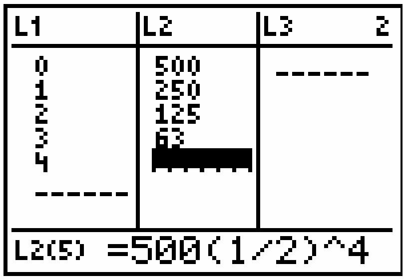


The M & M's with the M facing up are eaten (We feel good, but also "If you eat a candy bar right after a learning experience, it can enhance your memory of the experience." (Thompson & Madigan, 128)), and we toss the remaining on the table. We now have about 125 M & M candies on the table. How many do we expect to have the M facing up?
Students say: about 63 You say, "How did you get that?"
Students say: 125(1/2) You say, "But where did the 125 come from?"
Students say: 250(1/2)(1/2) You say, "But where did the 250 come from?"
See the edit line for the student answer.

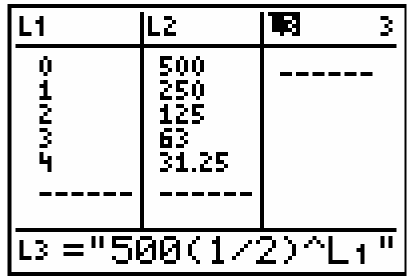


(Note: There are now 3 factors of ½)

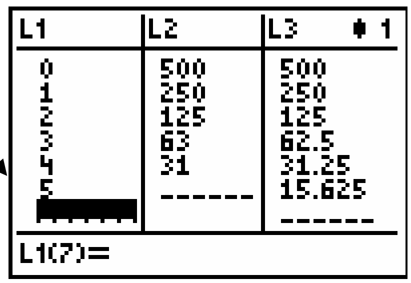
At this point most of the class recognizes the pattern (see Ackerman, 56-57) and you are ready to generalize and are prepared for the introduction of symbols which includes the concepts of variable, algebraic expressions, and modeling – see below.



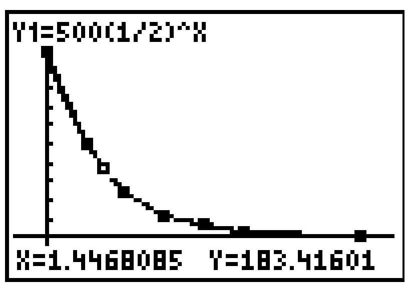
and then:



In L1, you can enter new values to see the power of abstract symbols – what a great discovery!



The values generated by the symbols are associated with the values the class generated. Thus, the symbols are now associated to data relationships. And as we see in the next graphic, these symbols are associated with the graphical representation of data relationships. Upon further investigation (the next step in the process – but at a later time because we want to distribute learning over time), we will see that one (or more) of the function parameters controls the increasing and decreasing behavior. But for now, we have accomplished the task at hand – demonstrating that data relationships (functions) can be represented symbolically as well as numerically and graphically.



The process of developing symbolic form with the list editor can be used for a variety of functions such as linear, quadratic, and rational – in addition to exponential. In every case, we use pattern building and guided discovery to create a model. We do not use regression or English-math memorization because the brain is looking for patterns, even though "For the most part we are not aware that we're constantly completing patterns, …" (Hawkins, 74). It is not unusual for students to discover a pattern after 2 – 3 iterations. A caution needs to be mentioned here because we know that two iterations may lead to recognizing an incorrect pattern. But this is why we embed the process as guided discovery with the teacher guiding the way through class discussion, or perhaps the concept is developed in a StudyCard lesson, using the same guidance.

We need to keep in mind that by making associations to the real world through contexts, we have also increased the likelihood our students will remember the concept. "Memory recall almost always follows a pathway of associations. One pattern evokes the next pattern, which evokes the next pattern, and so on." (Hawkins, 71) The mathematics being taught through real world contexts is easier to remember because of the many associations to life experiences. Understanding is enhanced by teachers building a pattern of a concept that the brain recognizes.

The Implementation Module Continues: Analysis of Geometric Behaviors of Functions

Students are now ready to learn the geometric behaviors (increasing/decreasing, max/min, rate of change, zeros, initial condition, when positive/negative, domain, and range) of basic functions. The time required to teach the geometric behaviors of basic functions can vary to a great extent by the audience, but mostly by the level of expertise of the teacher. Anecdotal evidence will perhaps show the variation. An experienced teacher typically requires eight or nine class days with remedial college students. But other college faculty may use more time because they try to teach more than behaviors. For example, one instructor upon teaching increasing/decreasing and constant rate of change decided that since these were behaviors of the linear function, he should teach the entire content set that is normally taught on linear functions – while in the middle of teaching function behaviors. Another example is a high school teacher who took about eight-ten weeks of class to teach behaviors. The problem was that she was not using the methods described in this article. Rather, she was teaching through memorization and demanded students to have a deep level of skill mastery. This is not the intention of the implementation module. The implementation module is to introduce the basic mathematical concepts, to prime the brain for later work (more on priming later), and establish associations, but not to acquire full skill mastery. Mastery of the behaviors comes later when each individual function is further studied as we distribute learning throughout the course.

As you saw earlier, we have already introduced increasing and decreasing. What is different now is that since we have introduced symbols to represent data relationships, we can now use symbolic forms of functions to analyze function behaviors, in addition to the contextual data relationships. At the same time, we can build on what we already know and make associations with new – and slightly higher level – content.

Why do we teach function behaviors? The major reason is that if we are going to use function concepts to teach algebra, students must know something about behaviors. For example, suppose your students have not studied function behaviors but you want them to use a graphing calculator to solve the equation $2x^2 + 41x = 115$. What do they do? More than likely they graph the function $y = 2x^2 + 41x - 115$ in the 10×10 (or decimal) window, find the zero of the function to be 5/2, and call it the root of the equation and then quit. This likely does not happen under the function approach because students have studied the behaviors of the quadratic function, know the number of roots (zeros) possible, and are familiar with the shapes of the graphs. When factoring trinomials, the process flows naturally since students have been taught how to find zeros and they have studied the connection between function parameters and zeros.

A second reason (of many) why we teach behaviors is that once students understand behaviors we can use, for example, constant rate of change and initial condition behaviors of the linear function to teach addition of polynomials. We can (and do) use the distributive property to teach addition and subtraction of polynomials too, but after we have used a function-based method. We need to make the underlying mathematical associations first. We need the associative cues students can use to recall the mathematics next year or later. We need to prime the brain to make the ideas available for more typical algebra. We want to capitalize on the innate visualization brain processing of mathematics. We need to benefit from the innate and learned number sense. We need to distribute learning over time. We need all these and other brain/mind attributes discussed later to help assure success with understanding. Most all concepts in algebra can be taught through function or behaviors of functions.

Below is a series of examples on how we might teach the zero behavior. The materials are presented here as pencil and paper activities with classroom formatting eliminated. But in the classroom, we use a mix of pencil and paper as well as StudyCard, Transformation, or Cabri Jr. electronic activities because we need to teach with multiple modalities so that students <u>correctly</u> remember what we teach. (Beversdorf)

The sample situations below may seem difficult to you, or not. If they do seem difficult to you, keep in mind that students have seen these functions before as data sets when they were asked about shapes and increasing/decreasing behaviors! We are continuing the process of building associations, and have included the final symbolic forms of the models of the situations – yet they have seen these before too in the section when we were developing symbolic form through pattern building.

<u>Example 1: Exploration</u>

1. A 1000-ml I.V. drip is being administered to a hospital patient at a drip rate of 2.5 ml per minute. The function that models the amount of I.V. fluid left is $Amount = -2.5t + 1000$, where t is time in minutes. When will the I.V. bottle have <u>no fluid left</u>?

2. A small car (1988 Camry) with a 12.8-gallon gasoline tank averages 32 miles per gallon driven. The function that models the amount of gasoline left in the gas tank is $G = -\frac{m}{32} + 12.8$, where *m* is the number of miles driven. When will the <u>gas tank be empty</u>?

3. A postal worker has 3224 pieces of mail to sort before it can be delivered. He can sort at a rate of 1.2 pieces per second. The function that models the amount of mail left to sort is $m = -1.2t + 3224$, where *t* is time in seconds and *m* is the amount of mail left to sort after *t* seconds have passed. When is there <u>no mail left to sort</u>?

4. A window washer in the Dallas - Fort Worth Airport has 873 windows to wash before she can take a break. She can wash windows at a rate of 1 window every 12 seconds. The function that models the number of windows left to wash is $w = -\frac{t}{12} + 873$, where *t* is time in seconds. When are there <u>no windows left to wash</u>?

5. On June 29, 1994, 15 Japanese beetles were sighted in Ed's red raspberry patch. Each day thereafter he observed 3 more beetles per day; if we assume the relationship is linear, the function that models the number of beetles in the berry patch is $B = 3t + 15$, where t is in days. When were there no beetles in the berry patch?

6. If you throw a ball straight upward with an initial velocity of 6 feet per second and it leaves your hand when it is 5 feet above the ground, the model of the height of the ball (assuming we ignore resistance to air) is $h = -16t^2 + 6t + 5$, where t is measured in seconds. When will the ball have a zero height?

Teachers using a traditional or "standards" curriculum (remedial or not) often view the first activity (above) as a set of exercise-type "word" problems, and fail to recognize it as a discovery teaching activity. That is, they do not view this as a teaching activity, but as a skill building activity. The same can be said of Examples 2 and 3 below. It is a matter of placement; if assigned after the teacher has "taught" the concepts, then these activities are summative in nature. But they were designed as teaching activities. In this context students are required to look for patterns and draw conclusions which requires them to think about the mathematics (see Examples 2 and 3), and they have several options for finding the answer to the questions. Langer refers to this as "mindful learning," and her research indicates we produce more creative students from this method. "An awareness of alternatives at the early stages of learning a skill gives a conditional quality to the learning, which, again, increases mindfulness." (Langer, 28) The process of teaching function behaviors provides an enriched teaching/learning environment – see more on this later – because we use a variety of teaching techniques and multiple mathematical methods. The visual attribute to learning has not been address yet in this paper; it will be developed later as a separate section. But, we have been using dynamic visualizations throughout the implementation module.

The next two examples are assigned without a context. Typically, mathematical concepts are developed in a context. But after there is conceptual understanding of the mathematics, we are able to move to non-contextual development of typical mathematical skills. That is, contextual situations are needed for introducing ideas from a neuroscientific perspective, but are not needed for content taught after it has been introduced contextually. In the examples below, you will still use pattern building as the teaching method. In Example 2, students quickly see the pattern that the zero is the opposite of the constant which leads to the generalized response in item 7 as being $-a$ as the zero of the function. This activity is simply a small part of the over-all development of the zero behavior that will over time lead to factoring and solving equations.

Example 2: Exploration

1. Find the zero(s) of $y = x - 3$	1. ______________
2. Find the zero(s) of $y = x + 2$	2. ______________
3. Find the zero(s) of $y = (x - 3)(x + 2)$	3. ______________
4. Find the zeros(s) of $y = x - 7$	4. ______________
5. Find the zeros(s) of $y = x - 3$	5. ______________
6. Find the zeros(s) of $y = (x - 7)(x - 3)$	6. ______________
7. Find the zero(s) of $y = x + a$	7. ______________
8. Find the zero(s) of $y = x - b$	8. ______________
9. Find the zero(s) of $y = (x + a)(x - b)$	9. ______________
10. Create any function that has a zero of 8	10. ______________
11. Create any function that has a zero of –5	11. ______________
12. Create any function that has zeros of 8 & –5	12. ______________
13. Create any function that has a zero of c	13. ______________
14. Create any function that has zeros of d & c	14. ______________

Example 3 is an activity that meets a misconception head on. Students are given the activity to convince them that looks can be deceiving. They may use the zero-finder on the graphing calculator to overcome the "looks" of a zero when in fact there is none. At the same time, this activity introduces the concept of geometric transformations. No reference is made to it, but this activity will be revisited when the topic is taught. This activity primes students for teaching transformations at a later time.

Example 3: Concept Quiz

1. $y = \lvert x-2 \rvert + 1$ has no zeros. Why?	1. ______________
2. $y = \lvert x-2 \rvert + 0.5$ has no zeros. Why?	2. ______________
3. $y = \lvert x-2 \rvert + 0.2$ has no zeros. Why?	3. ______________
4. $y = \lvert x-2 \rvert + 0.01$ has no zeros. Why?	4. ______________
5. $y = \lvert x-2 \rvert$ has a zero. Why?	5. ______________

6. $y=(x-2)^2+1$ has no zeros. Why? 6.____________________________

7. $y=(x-2)^2+0.5$ has no zeros. Why? 7.____________________________

8. $y=(x-2)^2+0.1$ has no zeros. Why? 8.____________________________

9. $y=(x-2)^2+0.01$ has no zeros. Why? 9.____________________________

10. $y=(x-2)^2$ has a zero. Why? 10.____________________________

Once students finish Example 1 from above, they have an idea of the meaning of a zero. It has several purposes. First, it moves students toward more traditional algebra; it primes students for factoring with zeros and solving equations with the zeros method; and through pattern building, it suggests another method for finding zeros through the parameter-behavior connection. That is, they recognize the pattern connecting parameters to zeros, so there is no need for the graphing calculator or pencil and paper in later work.

The three examples on zeros use pattern building through guided discovery; this concept is used throughout the module and later in the course. Perhaps we need to be reminded about power of pattern recognition from Nobel Laureate Gerald Edelman "… the brain's capacity to generalize is astonishing. … there are two main modes of thought-logic and selectionism (or pattern recognition). Both are powerful, but it is pattern recognition that can lead to creation, for example, in the choice of axioms in mathematics." (38, 147)

It may be appropriate here to make an observation relative to teaching delivered through e-format. You may be thinking that if you add a function implementation module to your course, you will not finish your required curriculum. But e-learning/teaching activities on the graphing calculator can be assigned <u>outside</u> of class. Most e-activities can be teaching in nature. If you teach your class while your students are on the bus trip home; or at home in the evening, you will typically be able to complete your full course. But <u>more importantly</u>, the function approach makes traditional algebra easier to understand because we use techniques (pattern building, contextual situations, visualizations, and priming) that are in concert with the way the brain functions, so you need less in-class teaching time on many concepts.

The Role of Priming

In the previous section, the word "prime" was introduced, and the concept of priming deserves a little more notice as a learning tool. It is not obvious to us, but considerable brain processing takes place at the unconscious level (as much as 98%) – including a learning and decision-making module. (Stanovich, 44), (LeDoux, 27) "The cognitive science literature is simply bursting at the seams with demonstrations that we do complex information processing without being aware of it, …"(Stanovich, 54). Stanovich lists over 20 research supported brain modules like the intuitive number sense module (44) that are active at the unconscious level, including modules for learning, counting, and estimating. At the unconscious level "TASS [The Autonomous Set of Systems] will autonomously be responding to stimuli, entering processing products in working memory for further consideration, triggering actions on its own, or at least priming certain responses, thereby increasing their readiness." (Stanovich, 49) This idea will set the stage for our brief discussion of priming.

Gladwell (10-11) describes an experiment designed by Antonio Damasio of the University of Iowa. It is a card game where drawing a card from the red deck results in a negative event (loss of money or a small gain), but drawing from a blue deck results in a more positive gain. The conscious side of the average brain took the drawing of around 80 cards to figure out the game, but only around 10 cards were needed by the "adaptive unconscious" side to start to show a favorite deck. The point is that the unconscious learns much faster than the analytical conscious. Further, "Explicit memory involves awareness, but priming does not, …" (Thompson & Madigan, 19) We learn without being aware of it.

Gladwell (23) continues by describing one experience used for priming. "'Thin-slicing' refers to the ability of our unconscious to find patterns in situations and behavior based on very narrow slices of experience." "… when our subconscious engages in thin-slicing, what we are doing is automated, [and] accelerated …" The implementation module provides a thin slice of experience from which the brain can draw throughout an algebra course to prime the conscious with ideas and direction. So adding it helps prime the brain which accelerates learning.

Let's consider the following example – it is not much different than most all of us have experienced when we have a modified "Aha!"

> "The physicist and biologist Leo Szilard made a similar point: 'Those insights in science that have led to a breakthrough were not logically derived from preexisting knowledge: The creative processes on which the progress of science is based operate on the level of the subconscious. Jonas Salk has forcefully articulated that same insight and proposed that creativity rests on a "merging of intuition and reason."'" (Damasio, 189)

Even though we may not be "thinking" about a problem, our senses or a thought on the conscious side of the brain may trigger a solution that had been processing in the unconscious. The unconscious can make ideas ready for processing by the conscious. Suppose you have a problem (exercise, project, exploration, etc.) that you need to solve. But the solution requires some mathematics you have never learned. We know that a very small percentage of people will invent the mathematics they need as part of the solution to the problem, and it may be entirely processed on the unconscious side. But let's suppose that the mathematics needed to solve the problem had actually been taught, but not with much mastery. In a case like this, the mathematics resides in the brain but is not readily available to be used by the conscious reasoning brain. This is a situation like what might happen under the function approach found in the implementation module. We put simple, but major ideas on function in the implementation module, and they now reside both on the conscious and unconscious sides, but they will be processed on the unconscious side and brought to consciousness when we teach the concept in class or in an activity. These basic ideas are then used throughout the algebra course and are available to prime student's analytical thinking when the more traditional algebra topics are taught using the function approach. So, when you are ready to use zeros to solve equations, students have already processed the idea and are "primed" to learn associated ideas. Priming puts ideas in the brain that are processed by the unconscious to make them ready for analysis by the conscious.

Once something learned has been stored in the neocortex, it is more available for the analytical side of the brain as described below.

> [a mathematician stares hard at a problem and says] "How am I going to tackle this problem?" If the answer isn't readily obvious she may rearrange the [problem] equation. By writing it down in a different fashion, she can look at the same problem from a different perspective. She stares some more. Suddenly she sees a part of the [problem] equation that looks familiar. She thinks, "Oh, I recognize this. There's a structure to this [problem] equation that is similar to the structure of another [problem] equation I worked several years ago." She then makes a prediction by analogy. "Maybe I can solve this new [problem] equation using the same techniques I used successfully on the old [problem] equation." She is able to solve the problem by analogy to a previously learned problem. (Hawkins, 185)

Hawkins argues that the neocortex is constantly looking for patterns through which it can "figure out" the world. It compares the new situation with stored patterns; when it finds a match, it understands the new situation. So, if the problem is written one way with no solution in mind, then he describes rearranging the problem – looking at it in a different light – so that the neocortex might recognize the new pattern that presents itself in the problem. One might argue that this process does not require the unconscious. However, "It is important to recall … [unconscious] processes pervades all functioning, and it cannot be "turned off" but instead must be overridden on a case by case basis." (Stanovich, 112) Even in cases where our students "know" what is needed to solve a problem, the unconscious is still priming the conscious – making it ready to be thought about. "Important overt behaviors can be affected by conceptual associations that are automatically triggered by TASS [the unconscious]." (Stanovich, 56) So we are back to the implementation module again. It becomes the knowledge base and priming base for algebra taught in the remainder of the course.

The Implementation Module Continues through Parameter-Behavior Connections.

In the natural progression of teaching from a function approach, parameter-behavior questions come to a head. I wonder why some lines are steeper than others? Why do some parabolas open up and others down? Are the rates of change of the branches of the graph of an absolute value function related? What causes the vertex to be where it is? Can't I find zeros without using a graphing calculator or pencil and paper? The answers to these questions are found by studying the parameter-behavior connection embedded within the study of behaviors. How do you teach parameter-behavior connections? One excellent way is to use guided discovery activities. Another is to include guided discovery exercises in homework. For example, below is a guided discovery teaching activity that is typically assigned to student groups either as pencil and paper or electronic form – before the ideas are taught in class.

Exploration Class__________Name________________________________

For each of the following functions, find the maximum or minimum, and specify the range.

	Maximum	Minimum	Range
1. $3\lvert x+2\rvert-5$	_______	_______	_______________
2. $5\lvert x-3\rvert+7$	_______	_______	_______________
3. $2\lvert x+4\rvert+3$	_______	_______	_______________
4. $-2\lvert x-3\rvert+6$	_______	_______	_______________
5. $-5\lvert x+1\rvert+4$	_______	_______	_______________
6. $-2.6\lvert x-5\rvert-7$	_______	_______	_______________

7. Given the absolute value function of the form $d\lvert x+e\rvert+f$, where d, e, and f are real numbers and $d \neq 0$, answer the following questions:

   a. What is the maximum or minimum value of the function?______________________________

   b. What is the smallest (or largest) number in the range of the function?_____________________

c. What number, *d*, *e*, or *f*, helps you decide if there is a maximum or a minimum?____________

Below is an example of a guided discovery exercise that is embedded within other "practice" homework exercises.

Find the zero for each function in Exercises 15 - 17. Secondly, find the domain of each function.

…

15. $3\sqrt{x+4}, 2\sqrt{x+2}, -3\sqrt{x+1}, -\frac{1}{2}\sqrt{x-1}, \frac{3}{8}\sqrt{x-3}, 15\sqrt{x-6}, 4\sqrt{x-a}$

…

One question you might ponder is whether we are asking students to "memorize" the parameter behavior connections. We could, but memorizing content means what was learned will not reside in memory as long as if it were learned through visualizations, associations and pattern recognition. "Memorization appears to be inefficient for long-term retention of information, and it is usually undertaken for the purposes of evaluation by others." (Langer, 72) The sample exercise combines practice as well as discovery and pattern recognition. By the end of the exercise, you have to figure that most students are able to find the zero and domain of the last few functions without the aid of a graphing calculator, provided the teacher (textbook) has primed them to look for patterns. In this example we associate the zeros with the *x*-axis. We appeal to the brain function of processing abstract mathematics through the visual system. We most certainly capitalize on the mind's ability to recognize patterns. At the same time, the proper use of technology attracts, and keeps, the attention of students. Further, new material comes to mind more easily because students were primed by the implementation module and it relates to the previous content through familiar associations. We may need to be reminded "… that you can easily understand mathematical concepts, provided they are presented in a familiar way." (Devlin, 119)

We may want to consider the reasonableness of an analogy.

> Suppose a person only speaks Chinese and you need to communicate with him/her. So you try using hand gestures and simple single English word commands. You point to items so as to convey ideas. You speak English slowly and repeat what you have said. You write simple English words.

But you must ask yourself if this attempt at communication is as understandable as if you spoke Chinese? Is there a good chance that there will be misinterpretations? Does the process add a layer of difficulty to the understanding of what you are saying? It makes sense to the author (at least) that communicating with the brain using its method of processing makes algebra more understandable and reduces the level of difficulty.

The Visual Brain

Common sense may lead us to the conclusion that if you can "see" the mathematics, it is easier to understand. Perhaps this is why Bert Waits and Frank Demana used the mantra "The Power of Visualization" with their textbooks that integrated the use of graphing calculators. One might wonder why it is that seeing a graph makes the connected symbols more meaningful and understandable. It turns out visualizations help us <u>understand</u> and <u>remember</u> what we are learning, as noted by Schacter "… after studying pictures along with the words, participants expect more from their memories." (103) At the same time, "Thanks to graphs, we primates grasp mathematics with our eyes and our mind's eye." (Pinker, 359)

The graphing calculator is the tool used to process visualizations of mathematics. As indicated below in the section on an enriched teaching environment, graphing calculator technology is essential. It is the tool that facilitates many of the activities that add to the novelty, the multiplicity of methods, the attention, and the associations used in the teaching process. There is more to the concept of visualizations than being able to see a graph.

Our minds are a product of many of years of evolution. But evolution is a slow process, and changes in our society are extremely quick by comparison. This means we are using brains that were designed for hunting, gathering and defending from our personal enemies. But the way we cope in a modern society is that the mind adapts already developed brain systems to function in new ways. For example, Pinker makes the point that "… we primates grasp mathematics with our eyes and our mind's eye. Functions are shapes (linear, flat, steep, crossing, smooth), and operating is doodling in mental imagery (rotating, extrapolating, filling, tracing). In return, mathematical thinking offers new ways to understand the world." (359) He goes on, "So, vision was co-opted for mathematical thinking, which helps us see [understand] the world." (360) As a result, the portion of the cortex devoted to remembering and processing visual information is disproportionately large. This is significant information as we try to decide how to teach algebra – through the assistance of visualizations or through pure symbolic processes. ~~In Pinker's book *How the Mind Works*, he devotes an entire chapter to explain how vision works on a cellular level so that he can make the statement that "vision was co-opted for mathematical thinking."~~ Since our minds process mathematics through the visual system, it makes sense that we can help our students understand mathematics by using dynamic visualizations in the teaching/learning process. Dynamic visualizations are preferable to static because they add the attention-getting quality of motion. Using dynamic visualizations to help us understand algebra has another added benefit. Both Langer and Schacter reference studies that show our memory of an event is better if we include visual information in the process, and that we easily forget items that do not carry the associated visual information. (Schacter, 103) (Langer, 42) At the same time, Schacter observes "Any attempt to reduce transience [one kind of memory loss] should try to seize control of what happens in the early moments of memory formation, when encoding processes powerfully influence the fate of the new memory." (34) So when these two ideas are merged, we find that <u>we must use visualizations at the beginning of any teaching lesson</u> if we want a better memory of what we teach; a less transience memory of the mathematics, and a better chance of understanding the mathematics!

Ackerman argues "because we have visual, novelty-loving brains, we're entranced by electronic media." (157) Mathematics educators may think of the graphing calculator as a tool to do mathematics, but our brain sees it as a device that provides novelty (through motion and content) and visualizations, which helps it understand mathematics and keeps it attentive. The function approach to teaching algebra capitalizes on the power of visualizations by always using visual representations of the algebra to be learned before, and/or in concert with using the symbolic and numeric representations.

Enriched Teaching Environment

An enriched teaching environment, in the context of teaching algebra, means teaching multiple methods for doing algebra with a variety of teaching tools, and using a variety of teaching methods. This idea is a natural fit for teaching from a function approach and using a graphing calculator. For example, we use and teach five methods for solving any equation – one of which is the pencil-and-paper method. We teach factoring through functions, dynamic graphing, and zeros of functions in bed with visualizations, guided discovery, and pattern building. All algebra content is first taught through some form or relationship to functions (to use associations and visualizations), and then traditional symbolic approaches are used. Because the approach requires the graphing calculator, we have many more opportunities for enriching the environment. Examples include electronic teaching of concepts and skills, reviewing, assessing through TI StudyCard e-activities, Texas Instruments LearningCheck™ e-activities, Cabri Jr. e-activities, and teaching understandings through dynamic visualizations. In addition, the option for variety in teaching through pencil-and-paper activities is greater because of the function approach. (Laughbaum, 2002)

The enriched teaching environment concept is based in animal studies. "Fred Gage and colleagues at the Salk Institute in La Jolla, California, placed adult mice in an "enriched" environment (one that resembles the complex surroundings of the wild more than the near-empty cages of the rats in the "non-enriched" environment). By the end of the experiment, the formation and survival of new neurons had increased 15 percent in a part of the hippocampus called the dentate gyrus. These animals also learned to navigate a maze better. … ." (Schwartz, 252) Human studies have since ensued with even better results. "… [R]egarding the effects of enriched environments on [human] brain structure, the results are credible and well established." (Byrnes, 184) (I have seen reports of studies that show a 25% increase in dendrite growth rate over using one teaching technique/method, but I can't relocate the source.) The least we can argue for is that when teaching in an enriched learning environment, we can expect more dendrite and neural growth in participating students than when we do not use multiple mathematical methods and technology. Since the function approach teaching environment does promote more neural growth, this would mean more synapses, more circuits, and thus, perhaps better thinking (dentate gyrus) and more capacity or improved memory.

In an email from the author to David Beversdorf (who has researched incorrect memory [learning]) in the Division of Cognitive Neurology at The Ohio State University Medical Center, the question was posed "In teaching, when you tie together voice, visual, and various activities like homework, guided discovery activities, and teaching activities on handheld devices, what are the odds of having a false memory about a particular topic you have taught with these techniques?" The reply was "[Y]es, evidence does clearly support that use of multiple separate modalities will decrease false memory effects." It appears that in addition to increases in neural strength, we also have increased the odds that our students will more correctly remember what we teach. This notion is confirmed by Byrnes "advocates of dual coding theory argue that people retain information best when it is encoded in both visual and verbal codes." (51) Daniel Schacter references research maintaining that "after studying pictures along with the words, participants expect more from their memories." (103) He also shows that "… more elaboration during encoding generally produces less transient memories." (27) As it turns out, the enriched teaching/learning environment is a significant teaching tool.

Using an enriched teaching/learning environment is relatively simple to implement when using graphing calculators and a function approach. At the same time, the traditional pencil-and-paper equation-solving symbolic approach seems not to embrace the concept. This may be part of the reason why many remedial algebra students have incorrect knowledge/understandings about algebra. When you hold incorrect memories of what you have learned, and are faced with remediated teaching/content, this may give rise to fear of mathematics and loss of memory.

Quite possibly one attempt made by publishers of math textbooks to engage the enriched environment concept is that they have added multi-colored page lay-outs, and learning/reference boxes in all kinds of formats. Yet, nothing on the printed page involves novelty, or any sense but the static visual. Instead of capitalizing on enriched environment learning, they have created visual overload.

The Payoff
When finished with the implementation module, function, function representation, and function behaviors are used in the teaching of more traditional topics. Student's manipulative skills may be weak, but this is remedied later in the course when symbol manipulation skills become important – in the function approach, we develop mathematical concepts first. Otherwise, the function approach would suffer from symbol manipulation without understanding – much like the traditional and "standards" approaches. We have only setup the basics in the implementation module. Typically, we do not want to rush into a full discussion of any topic until the appropriate time. (LeDoux, 106) Further, we need to distribute the learning of each concept throughout the course. (Thompson & Madigan, 92)

Below is an example of the payoff – assigned either in pencil and paper or electronic form – represented as teaching activities, not summative. That is, they are assigned as the learning method for factoring with no lecture/discussion preceding them. You will note the use of $f(x)$ notation which is an indication that the activity does not immediately follow the implementation module. We do not need to use a context since the concept of zero has been introduced in a context, so students know the meaning of a zero. We are simply using the zero concept to teach students how to factor. Like most all content, we start with something students already know about function, to make associations. Students use the visual representations to find the zeros, so that they are more likely to understand and remember the factoring ideas. Our students have been primed with all the needed background mathematics in the implementation module. Finally, we use a variety of pencil and paper along with electronic activities so that students will remember correctly what is being taught.

<u>Teaching Factoring through Guided Discovery Using Pattern Building:</u> (Please note that spacing has been removed for publishing purposes.)

Exploration 1 Class__________________Name____

1. What is the zero of the function $f(x) = 2(x-3)$ 1.__________________________
2. What is the zero of the function $g(x) = 2x-6$? 2.__________________________
3. How are functions *f* and *g* related? 3.__________________________
4. What is the zero of the function $f(x) = -4(x-3)$? 4.__________________________
5. What is the zero of the function $g(x) = -4x+12$? 5.__________________________
6. How are functions *f* and *g* related? 6.__________________________
7. What are the zeros of the function $f(x) = (x+1)(x-3)$? 7.__________________________
8. What are the zeros of the function $g(x) = x^2-2x-3$? 8.__________________________
9. How are functions *f* and *g* related? 9.__________________________
10. What are the zeros of the function $f(x) = (x-2)(x+2)$? 10.__________________________
11. What are the zeros of the function $g(x) = x^2-4$? 11.__________________________
12. How are functions *f* and *g* related? 12.__________________________
13. If the zeros of $f(x)$ are –1 and 3, create one possible $f(x)$. 13.__________________________
14. If the zeros of $f(x)$ are –4 and –2, create one possible $f(x)$. 14.__________________________
15. If the zero of $f(x)$ is 5, create one possible $f(x)$. 15.__________________________
16. If the zeros of $f(x)$ are –4, 2, and 1, create one possible $f(x)$. 16.__________________________
17. If *d* and *e* are the integer zeros of a quadratic function $f(x)$, create one possible $f(x)$. 17.__________________________

Exploration 2 Class__________________Name____

1. What are the zeros of the function $f(x) = (2x-1)(x+3)$? Express them as reduced fractions. 1.__________________________
2. What are the zeros of the function $g(x) = 2x^2+5x-3$? Express them as reduced fractions. 2.__________________________
3. How are functions *f* and *g* related? 3.__________________________
4. What are the zeros of the function $f(x) = (3x-1)(2x+5)$? Express them as reduced fractions. 4.__________________________
5. What are the zeros of the function $g(x) = 6x^2+13x-5$? Express them as reduced fractions. 5.__________________________
6. How are functions *f* and *g* related? 6.__________________________
7. What are the zeros of the function $f(x) = (2x-3)(x+2)$? Express them as reduced fractions. 7.__________________________
8. What are the zeros of the function $g(x) = 2x^2+x-6$? Express them as reduced fractions. 8.__________________________
9. How are functions *f* and *g* related? 9.__________________________

10. What are the zeros of the function $f(x) = (3x-2)(2x+3)$? Express them as reduced fractions. 10.______________________
11. What are the zeros of the function $g(x) = 6x^2 + 5x - 6$? Express them as reduced fractions. 11.______________________
12. How are functions f and g related? 12.______________________
13. If ½ and 3 are the zeros of a quadratic function $f(x)$, create one possible $f(x)$ containing integer parameters. 13.______________________
14. If $\frac{2}{3}$ and -3 are the zeros of a quadratic function $f(x)$, create one possible $f(x)$ containing integer parameters. 14.______________________
15. If $\frac{2}{3}$ and $-\frac{1}{4}$ are the zeros of a quadratic function $f(x)$, create one possible $f(x)$ containing integer parameters. 15.______________________
16. If $\frac{a}{b}$ and $\frac{d}{e}$ are the zeros of a quadratic function $f(x)$, create one possible $f(x)$ containing integer parameters. 16.______________________
17. Describe in your own words any connection you see between the zeros of a function and the symbolic form of the function.

Exploration 3 Class______________ Name______________________________

In the first two explorations, you learned more about the connection between function parameters and the related zeros of the function. Below is a quick review and then a continuation of the exploration.

1. Find the zeros of the function $y = (2x + 1)(x - 3)$. 1.______________________
2. Find the zeros of the function $y = 2x^2 - 5x - 3$. 2.______________________
3. Why are the zeros the same for $y = (2x + 1)(x - 3)$ and $y = 2x^2 - 5x - 3$? 3.______________________
4. Find ***any*** polynomial whose zeros are −5 and 5. 4.______________________
5. Find ***any*** polynomial with integer parameters whose zeros are $-\frac{4}{5}$ and 3. 5.______________________
6. Based on what you learned in the first two explorations, write the function $y = x^2 + x - 2$ another way using the zero-parameter connection. 6.______________________
7. Based on what you learned in the first two explorations, write the function $y = x^2 - 4$ another way using the zero-parameter connection. 7.______________________
8. The function $y = 2x^2 - 5x - 3$ can be symbolized another way. Write it using other symbols with integer parameters. 8.______________________
9. Why do you think the function $y = x^2 + 4$ cannot be written in different symbolic form through the zero-parameter connection? 9.______________________
10. Why do you think the function $y = x^2 + 2x + 4$ cannot be written in different symbolic form with integer parameters? 10.______________________
11. When you re-write a function like $y = x^2 + 3x - 28$ as $y = (x + 7)(x - 4)$, we say you are re-writing in factored form. Or we say you are factoring. For each of the following functions (expressed as quadratic expressions), re-write them in factored form. That is, factor them.
    a. $3x^2 - x - 2$ a.______________________
    b. $x^2 - 9$ b.______________________
    c. $20x^2 + 33x - 36$ c.______________________

By the time students have finished with these three electronic or paper activities (usually assigned outside of class), they are likely well on their way to learning to factor via the pencil and paper method, and they have likely mastered the function-based method. The experience of learning to factor through a function approach is rich in associated mathematics, and is not a disjointed process as it is in the traditional pencil and paper traditional curriculum. The richness of the method builds lasting memories of the mathematics learned.

Teaching the "Less-Than Property" for Absolute Values: A Classroom Discussion
A traditional option (when not using a function approach) is to state the property and then use it in several examples, followed by extensive practice – to help memory.

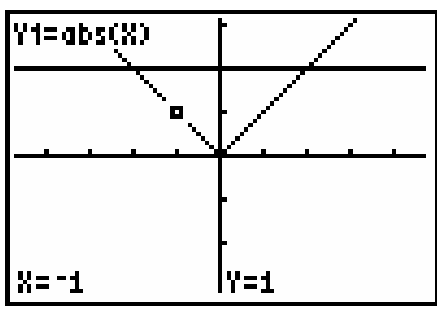
$$\text{If } |x| \le a \text{ for some positive number } a, \text{ then } -a \le x \le a.$$

But the problem is; what other mathematics is it associated with when presented in this manner? When were students primed for this topic? What are the associative cues for recalling this property? When teaching the less-than property by declarative statement and followed by examples and practice, where is the enriched environment? How do we help the brain use its visual processing abilities of abstract mathematics? What is used to procure the attention of the student? How is distributed learning being implemented?

We cannot count on rote practice for long-term memory retention nor understanding. Ellen Langer, in her book *The Power of Mindful Learning*, argues that "Memorization appears to be inefficient for long-term retention of information, … ." (72) She also does not have much hope for future performance of our students when learning through rote: "Learning the basics in a rote, unthinking manner almost ensures mediocrity." (Langer, 13) In the past, we may have assumed that assigning extended homework would cause student learning. We thought that it would take considerable practice to learn the mathematics. We thought that practice caused increased synaptic strength, which meant our student would remember the mathematics longer. While practice does increase synaptic strength, Eric Kandel found that: "Our studies showed dramatically that in circuits modified by learning, synapses can undergo large and enduring changes in strength after only a relatively small amount of training." (205)

Thinking from a function approach, we know that technology and the various contexts used in the implementation module with absolute value data relationships have gotten their attention and primed them for this mathematics. Students have analyzed all representations of absolute value functions. Given $y = d|x+e| + f$, they know what behaviors the *d*, *e*, and *f* parameters control. They have traced on the graph to make connections between representations and involved their learned number sense. They have associated absolute values with various real-world contexts, and students have made parameter-behavior associations.

In teaching from a function approach we <u>start</u> with the graph (or table) of the function $y = |x|$ and the graph of a positive constant function like $y = 2$, and build a pattern leading to the property.



What do we discover about $|x|$ as we trace or scroll back and forth between $x = -2$ and $x = 2$? It is that $|x|$ is always less than or equal to the positive number 2. Is this pattern true for $|x| \le 3$ or $|x| \le 1.7$ or $|x| \le 19$?

Hmmm, it seems that if $|x| \le a$ then $-a \le x \le a$. Do we also learn something about *x* when $|x| > a$? After the above guided discovery discussion is the time for formalizing the property with abstract symbols, followed by "thinking required" practice. Langer describes learning when students are required to think about what they are doing or are assigned work designed to make them think – as opposed to work on rote memorization. She says, "… we found that the students who did not rely on memorization outperformed the others on every measure …" (78) The guided discovery discussion described above requires thinking and reasoning. It also requires technology and basic knowledge of function behaviors. "The richer, more varied, and more challenging the experiences, the more elaborate the neuronal circuits." (Restak, 32) Elaborate neural circuits have more embedded associations and the information stored is more likely to be remembered because of this basic neural functioning.

Summary and Conclusion
This article makes the argument that algebra should be taught through a function approach implemented with a graphing calculator so that we can seamlessly capitalize on the brain's normal functioning. That is, textbooks need to integrate the concepts presented in this paper so that teachers can use the books without considerable preparation time needed if they are not integral. This article made the following arguments:

- We remember algebra longer and better by using associations – made through function permeating the content. That is, students are more likely to remember the mathematics taught because we capitalize on associations integral to a function approach and contextual learning.
- Learning is simpler, faster, and more understandable by using pattern building as a teaching tool. Almost all of the pencil and paper activities, e-teaching activities, and class discussions use pattern building. We build the patterns so that students can recognize the pattern and make the desired mathematical generalization – a natural function of the brain.
- Students cannot learn if they are not paying attention, so the graphing calculator is used to draw attention to the mathematics through its basic functionalities **including** the app software. At the same time, the function approach gets student attention because it is novel.
- Visualizations are integral to the function approach to assist with understanding and memory. In the function approach visualizations are used first before any symbolic discussion. This greatly increases the likelihood that students will remember the mathematical concept being taught. Visualizations are a basic catalyst to understanding, without them, we must work harder to understand.
- Considerable brain processing takes place in the unconscious side of the brain, including a learning module. To make this processing possible the brain must be primed. The implementation module and teaching e-activities primes the brain for the mathematics that follows, and this makes learning faster.
- The enriched teaching/learning environment promotes correct memory of math content. The wide variety of teaching activities facilitated by the function approach provides the enriched environment. The enriched teaching/learning environment reduces the brain function of habituation which left unchecked causes reduced memory.
- Contextual situations (represented as functions) provide meaning to the mathematics learned. Attached meaning allows students to function at a higher cognitive level, and provides associative cues for long-term memory. Mathematics taught without meaning creates memories without meaning that are quickly forgotten.
- Distributing learning of any concept over time has clear and positive benefits for promoting long-term memory of the concept taught. The function approach as described in this paper allows for a seamless integration of this tool.

Since the 1990's (the decade of the brain), neuroscientists and cognitive psychologists have made great advances in learning how the brain functions, in part because of the invention of the MRI, the fMRI, with considerable research since the 90's. They, in fact, have learned enough that those in education must start incorporating what is known about brain/mind functioning into the class room. In the case of teaching and learning of algebra, we must move to using a function approach with the right tools.

At the same time, we are not losing any of the traditional content. If fact the integrated use of function provides a much richer algebra curriculum. We make considerable connections within the algebraic content and to the world outside the classroom. The use of contexts promotes conceptual understanding and helps students to realize that mathematics is a humanistic discipline that is connected to the real world.

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NEED DESCRIPTION HERE

Comments from Current Users:
"I did a little within school comparisons using the EMPT scores and found that the students in *Foundations of College Math* did significantly better than other students in our Algebra II classes. Our *Foundations of College Math* students came from Algebra II classes where they failed, dropped the class or didn't have the confidence to take pre-calculus."

Tim Voegeli, Ohio

"The book is wonderful. The kids keep asking me where you get all those wonderful facts ... (the most recent favorite: 'there is enough TNT in the world for each person to have 10,000 pounds')."

Kayla Pinnick, Ohio

"We are using your series *Foundations for College Mathematics* as a senior course for those students going to college but not in a math related field. At this time, I am very pleased with the format of the textbook and the exercises at the end of each section. I like the function approach and believe it has helped my students."

Mary Inmon, Ohio

"Your book is a big adjustment for everyone – it's just a different approach. But, so far, everyone loves the way it's done."

Julie Merrill, Texas

"The students are doing amazingly well, and I credit part of that to a well-written text."

Matt Gaston, Ohio

"I visited classrooms and talked with both teachers and students. They are still pleased with the *Foundations* curriculum."

Peggy Ginn, Math Coordinator, Ohio

"For awhile now, the struggling math students in our school have found great success using the function approach."

Jeff Dabney, principal, Rocksprings, TX

Current Table of Contents:
Attached later.

Current Users:
There are currently 75 adopters of *Foundations*. Five of these are in colleges and the remainder in high schools. Of the colleges, one uses it for an independent study class, two for an algebra class; the other two are using it in teacher prep programs. *Foundations* was also used by all instructors of Intermediate Algebra at the author's home college during the development phase. In the past, *Foundations* was promoted only in Ohio – as Red Bank Publishing is the self-publishing company of the author – also based in Ohio.

Ancillaries:
The first is a 400-page student workbook with the following types of activities:

Explorations:
Explorations are of two types. A few are designed for the day before a topic is discussed in class. They are somewhat like guided-discovery exercises. Just like some of the explorations are designed for before you formally teach the related topic, others have been designed for after the formal discussion. If assigned at the wrong time, some will become exceedingly difficult and others may take what was a challenging assignment and turn it into a simpler and low-level thinking task. An exploration that asks students to solve the equation $2[x - 5(3x + 4) - 6] + 6[4 + 7(9x - 1) - 2(2x - 6)] = 0$ is to be assigned right after teaching students to solve linear equations (Section 6.1) – not before the topic is taught. Whereas an exploration something like in Section 2.1 asks students to predict the number of inmates in the US in the year 2000, given a numeric history of the prison population. This is assigned before the topic is taught. The intent is to see if they can use their own ideas to solve this problem. Most explorations require a graphing calculator, a few don't. The author used almost all of the explorations as group assignments; giving them to students as they left a class and collecting them at the beginning of the

next class. The title "Explorations" is somewhat descriptive of the kind of work the students will be doing. That is, many times students must explore on the calculator to answer the question.

The explorations are an integral part of the overall assessment tools used with *Foundations for College Mathematics*. Assessment tools like the explorations measure a totally different kind of learning than do skill-based midterms. Explorations typically use pattern building embedded in guided discovery.

Concept Quizzes:
Just like the explorations, some concept quizzes are to be assigned either before a topic is discussed or right after. Many of these are like guided discovery exercises and they would be assigned before the topic is taught. What is unique about the concept quizzes is that many ask students to do something they probably don't do much – be creative. These quizzes contain questions like "If $(f+g)(x)=2x+7$, develop any two functions $f(x)$ and $g(x)$ whose sum is $(f+g)(x)$." Or "Create any function that has a domain of [3.6, ∞)." The concept quizzes usually require a graphing calculator. The quizzes investigate a variety of concepts related to the mathematical topics in *Foundations for College Mathematics*.

The concept quizzes are an integral part of the overall assessment tools used with *Foundations for College Mathematics*. Assessment tools like the concept quizzes measure a totally different kind of learning than do skill-based midterms and the explorations that require "exploration" and a little tenacity. Concept quizzes typically use pattern building embedded in guided discovery.

Investigations:
The investigations usually take a situation or single idea and ask a multitude of questions about the situation or idea. They are assigned after a topic has been developed in class. They usually require a graphing calculator. The intent of the investigations is to require a thorough analysis of a topic or idea. For example, Section 2.1's investigation asks 14 questions about the electricity charge data from the North Carolina Public Utilities Commission. Or the Section 4.4 investigation has 26 questions about gasoline usage.

The investigations are an integral part of the overall assessment tools used with *Foundations for College Mathematics*. Assessment tools like the investigations measure a totally different kind of learning than do skill-based midterms and the explorations that require "exploration" and a little tenacity, or the concept quizzes that ask students to be creative. Investigations typically use pattern building embedded in guided discovery.

Writing Mathematics:
These assignments are typical writing assignments like found in many reform textbooks. Just like all of the above tools, the writing materials are intended as another method for assessing other facets of student understanding. While all of these tools are used as assessments, many are used to enhance understanding and to promote learning. Many of the other assessment/teaching tools listed above also require some writing. As do the modeling projects described below.

Modeling Projects:
The modeling projects require exploration, conceptual understanding, tenacity, and a graphing calculator. They assume some knowledge of the connection between function behavior and function parameters and/or they require the use of geometric transformations to create the mathematical model of real-world data. A study of statistics is not assumed. Students typically may require 2 to 10 hours to finish a modeling project.

The modeling projects provide an opportunity to ask students to apply the mathematics that is in the text. Mathematics such as using the relationships between the parameters in a function and the behavior of the function, geometric transformations, arithmetic operations of functions and the resulting change in the domain, behavior near the zeros of functions, etc. Students must recognize the shapes of the basic elementary functions. They must know how arithmetic operations on functions changes the geometry of the graphical representation of the functions. The main goal of the modeling projects is to find a symbolic representation of any function that models the data.

Just as an engineer, physicist, business person, or astronomer may solve a problem by following a prescribed procedure, your students are directed through this problem solving process by being asked a series of questions about the data. Once they have solved the main problem and have developed the symbolic representation of the function (mathematical model), they are asked to defend their solution by explaining the limitations of their model. This includes giving a situation when their model does not apply. They must explain their thinking on how they developed the model. Students are given the opportunity to use their model when asked questions about data not available in the given information. They are also asked to conjecture on what type of professional person might use their model. And finally, they must identify the references and resource person(s) used to help them in the problem solving process.

Class time may be used to work on the projects or they provide a good assignment outside of class. Depending on the level of mathematical sophistication of students, the projects may take from one to four hours. The modeling projects provide another tool for assessment that measures still other traits of the math student.

The remaining ancillaries are:

Data Sets:

All data used in *Foundations* is available in TI-84 or TI-83 Plus calculator programs, grouped by chapters as TI calculator group files.

e-Activities:

A series of 48 StudyCard stacks on algebra topics that can be used as a "Power Point" like presentation, or assigned to students to be executed right before (or at the beginning) class. They are to be processed in a group of two students. A subset of the 48 activities is to be used as a summative assessment tool that can be used in the place of a quiz. See the full description in the supporting documents below.

Miscellaneous:

A 100-page document that addresses how selected homework exercises can be solved (a teacher manual). Even Answer Key/Chapter Test Key and a two semester suggested course outline.

Competition:

Remedial/developmental algebra texts have yet to use a true function approach (other than one by an author who copied *Foundations* ideas and structure for a major publisher). Several others claim to use the function approach, but it simply means moving the chapter on functions from eight to two (for example). Of the five or six claiming to use a function approach (or to satisfy standards-based mathematics requirements) they never use function concepts and behaviors of functions to teach mathematics. Further, functions rarely appear anywhere else in the text. That is, no one is using function and function behaviors to teach mathematics. No one is using function and function behaviors to do mathematics such as polynomial addition & subtraction, reducing rational expressions (function), laws of exponents, factoring, equation solving, systems of equations, inequalities, properties of inequalities, definitions, etc.

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Preface to Students and Instructors

This text contains terminology, content, and algorithms that may not be found in a traditional textbook because it is the author's intention to break from tradition and prepare students for the mathematics needed in a modern society. Further, as learning progresses, terminology may change to reflect new understandings. The author also recognizes that mathematics is learned by understanding, not by memorization. Developing mathematical ideas in the context of a situation helps students understand mathematics; at the same time, students understand that mathematics relates directly to the world.

The use of this text requires a graphing calculator with function notation; the calculator will be used as a tool to help us understand mathematical concepts and perform mathematical algorithms. Technology, which offers students and teachers a variety of methods for solving problems, is used in this text to explore mathematical ideas, and it changes what is considered important. Many traditional mathematical topics have diminished importance. This text offers content that is important to students directly entering the work force and to students continuing the study of mathematics and science. Many of the exercise sets offer questions that are engaging and demand higher level thinking.

Students have the responsibility to read and study the text in order to learn. Students have the responsibility to view the exercise sets as questions that are to stimulate thinking, not questions that are designed to encourage memorization of facts. Students have the responsibility to recognize that the exercise sets contain questions to help them formulate their own ideas about mathematical relationships.

There are limited references to calculator keystrokes; when they are included, they are for theTI-83 or TI-84. With slight modification, the keystrokes will also work on other calculators.

The teaching features of this text include:

- Technology integrated throughout to enhance the learning and teaching of mathematical concepts.
- Technology integrated throughout to provide options for performing mathematical algorithms.
- Mathematical concepts introduced in the context of real-world situations.
- Thorough analysis of applications.
- Guided discovery exercises.
- Reduced emphasis on the use of symbol manipulation and increased emphasis on the use of function as a central theme.
- Three methods (numeric, graphic, and algebraic) of representing a function.
- Distributed learning has been incorporated in this text. For example, the idea of function is introduced in Section 2.1 as a data relationship. The intuitive idea of function is further developed in Chapter Three by looking at algebraic expressions that model data relationships introduced in Chapter Two. Chapter Four continues with formal function notation, and functions are used in equation and inequality solving in Chapter Six. Chapters Seven, Eight, Nine, Ten, Twelve, and Fourteen offer analysis of individual function types.
- A variety of methods for solving problems are encouraged. Students are encouraged to explore on their own.
- Higher level thinking skills are encouraged through projects, open ended questions, and concept questions.
- Sample problems are checked numerically or graphically and students are encouraged to do likewise.
- Exercise sets contain low-level difficulty problems that prepare students for future topics.
- Exercise sets contain high-level difficulty problems that review topics previously taught.

- Exercise sets contain writing questions.
- Exercise sets contain concept questions.
- Exercise sets contain exploration problems. Many of the explorations can be used for group work. Many can be used as portfolio exercises.
- Exercise sets contain open ended questions.
- Mathematical words are defined by bold text.
- Extended laboratory projects on modeling are included with the text.

Neuro and cognitive science research has provided considerable information about how the brain functions. This textbook capitalizes on this research through the implementation of the cognitive processes of associations, pattern recognition, attention, visualizations, priming, meaning, and the enriched teaching/learning environment.

- We remember algebra longer and have better memory by using associations – made through function permeating the content. That is, we are more likely to remember the mathematics taught because we capitalize on associations made through using a function approach.
- Learning is made simpler, faster, and more understandable by using pattern building as a teaching tool. In the function approach used in this text, almost all of the pencil and paper activities, e-teaching activities, and class discussions use pattern building to reach a generalization about a concept or skill.
- We cannot learn if we are not paying attention. The graphing calculator is used to draw attention to the mathematics through its basic functionalities **including,** various app software.
- Without visualizations, we do not understand or remember the mathematics as well. In the function approach visualizations are used <u>first</u> before any symbolic development. This greatly increases the likelihood that we will remember the mathematical concept being taught.
- Considerable brain processing takes place in the subconscious side of the brain, including a learning module. To make this processing possible for our purposes, the brain must be primed. The function implementation module (Chapters Two and Three) and early e-learning activities prime the brain for all the algebra that follows.
- The enriched teaching/learning environment promotes <u>correct</u> memory of math content. The wide variety of teaching activities facilitated by the function approach provides the enriched environment.
- Contextual situations (often represented as relationships) provide meaning to the algebra learned. Algebra taught without meaning creates memories without meaning that are quickly forgotten.

Algebra approached through function, as in this textbook, has reordered the content and capitalized on function concepts to <u>develop understanding, long-term memory, and skills</u>.

This text has been published as a preliminary edition, as a revised preliminary edition, a first edition test, and now as a second edition. It has been class tested over a ten year period and has been revised and edited no less than twelve times. It has been reviewed by over twenty-five reviewers.

Philosophy and Use of
Foundation for College Mathematics* 2*e - Laughbaum

Chapters Two and Three contain material that is intended as an introduction to several basic functions. The other chapters continue the distributed learning spiral with more advanced material on these and other functions. Although Chapters Two and Three introduce students to various functions in an intuitive fashion, you will also recognize that the included topic of "behavior of functions" is an extremely important part of the curriculum and is used throughout the text *Foundations for College Algebra 2e*. It is wise to use as many as three weeks on the content in Chapters Two and Three. Time spent here will save time two to three-fold in later material. Further, not spending three weeks on Chapters Two and Three will cause difficulties for students later in the book. For the full learning effect, materials from *Leading Discussions, Explorations, Concept Quizzes, Writing Mathematics, Investigations, and Modeling Projects for Foundations for College Mathematics* should be utilized, starting in Chapter One and fully implemented in Chapter Two and beyond.

The terminology in this companion work to *Foundations for College Algebra 2e* may be slightly different from that in other textbooks in four ways: (1) In order to apply it to real-world situations, the topic of *domain* has been split into two parts. If mathematics is used without reference to functions representing anything except real numbers, the domain is called the *normal domain*. However, if a function is used to represent a relationship in the real world, the domain will quite often become a subset of the normal domain and is called the problem domain. For example, the function $S(c) = \dfrac{300}{6c + 300} \cdot 100$ has a normal domain of all real numbers except −50. But, if $S(c)$ is used to model the strength of a coffee-cream mixture where c represents the number of 6-*ml* cream containers added to 300-*ml* of coffee, the normal domain is of little use. The domain that makes sense here is not the normal domain but a problem domain such as the positive integers. (2) This manual uses the notation $S(c) = \dfrac{300}{6c + 300} \cdot 100$ with the notation $\dfrac{300}{6c + 300} \cdot 100$ interchangeably as the symbolic representation of a function. The first notation is the symbolic form of the ordered pairs of numbers $(c,\ S(c))$, and the second is the symbolic form of the set of ordered pairs of numbers $\left(c, \dfrac{300}{6c + 300} \cdot 100\right)$. Both $S(c)$ and $\dfrac{300}{6c + 300} \cdot 100$ represent the strength of the coffee and c represents the number of cream containers added to the coffee. (3) Emphasis is placed on rate of change; therefore, use of the word *slope* has been minimized because it is used little in the real world. (4) The different representations of a function relationship are called symbolic representation, numeric representation, and graphic representation. This is used because the word "representation" ties all three together. This connects all three to the concept of function; unlike the words rule, table, and graph.

There is no answer key to *Leading Discussions, Explorations, Concept Quizzes, Writing Mathematics, Investigations, and Modeling Projects for Foundations for College Mathematics*. The reason is twofold. First, most of the questions are open-ended, meaning that there are many correct responses. Second, sample answers encourage students and instructors to look for answers similar to the answers in the answer key. This stifles creativity. Creating your own answer from your own mind is a highly rewarding experience and causes a desirable higher level of thinking. The spark that can cause this higher level of thinking may come from a group setting. Working toward a "back-of-book" answer impedes the process.

Similarly, there are no sample answers to writing questions. The quality of written answers is a function of the mathematical maturity of the student. Many beginning and intermediate algebra students may not use the correct terminology, but they will get the idea across using the words they know.

The materials in *Leading Discussions, Explorations, Concept Quizzes, Investigations, Writing Mathematics, and Modeling Projects for Foundations for College Mathematics* may be duplicated for classroom use only. They may not be reproduced in any format for any other purpose without the written consent of the author.

Below is a listing of features found in *Foundations for College Mathematics 2e*. These features have been recommended by a variety of reform documents. Also included are reasons for the features and how they can be used.

<table>
<tr><th>FEATURES</th><th>RATIONALE AND USE</th></tr>
<tr><td>Technology integrated throughout to enhance the learning and teaching of mathematical concepts.</td><td>
Every section uses technology to enhance either teaching or learning or both. As early as Section 1.1, the calculator is used to develop an argument for justifying the field properties. Calculator logic statements allow for the demonstration of the use and misuse of the properties of equality and inequality.
Section 1.2 continues the use of technology by “looking” at data pairs through using stat-plot. Students encounter related data in numeric form outside the math classroom; now, they can take this data and discover that it many times has a very definite “shape.” Students are therefore immediately introduced to the concept of function using this simple approach.
The purpose of Chapters Two and Three is to set the groundwork for learning about connections between function behavior and function parameters. This is accomplished by using the graphical or numerical representations of primary functions. Guided discovery of the behavior-parameter connection cannot be completed without technology. Chapters Two and Three are also intended as the time for students to become familiar with the calculator. That is, you should not take class time to “teach” the calculator as an end goal. Calculator mechanics must be taught in the context of mathematics. This is a minor, but necessary, goal for Chapters Two and Three.
Of course, since students learn about zeros of functions in Chapters Two and Three, factoring can be developed by studying the connection between zeros and linear factors of polynomial functions in Chapter Four.
There are numerous uses of technology integrated throughout the text. A few uses are to:

do arithmetic,
find intercepts and slopes of linear functions,
study behaviors of functions,
analyze function operations from a graphical perspective,
solve all kinds of equations and inequalities,
model collections of real-world data pairs,
confirm answers to algebraic simplification,
develop and confirm algebraic properties,
discover changes in the domain of algebraically simplified expressions,
find extraneous roots caused by solving equations with pencil and paper,
evaluate trig functions,
solve systems of equations and inequalities,
study function notation, and

</td></tr>
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<table>
<tr><td></td><td>• analyze compositions of functions.</td></tr>
<tr><td>• Technology integrated throughout to provide options for performing mathematical algorithms.</td><td>

Students must be given options for performing mathematical algorithms because this is a good model for problem solving in general. Many times, there are multiple methods for solving problems. Further, students must be responsible for selecting an algorithm. This too, models expectations in other arenas. From a pedagogical standpoint, a student may not understand one algorithm, but if they have options, they should understand another and be able to use it.

The primary example is in solving equations. Suppose you assign the following equation to be solved $(x-2)^2 - 3|x+1| = 4$. This is certainly formidable to the student who only knows how to solve equations with pencil and paper. Chapter Six illustrates four other technology-based algorithms. Three of these: zeros, intersection and numerical work quite well with this equation. On the other hand, asking students to solve the equation $\pi x = 3$ ought to cause them to respond with pencil and paper or they should solve it mentally.

Suppose students must evaluate $\left(\frac{-b}{2a}\right)^2$ and produce a value in rational form. Pencil and paper will work fine, but students quite often make mistakes when evaluating it. Chapter Nine offers a technology-based method that will probably be less prone to error. The instructor should use both methods so students see they have a choice.

Examples 8 and 9 from Section 9.5 demonstrate how multiple methods are modeled in the text.

Example 8: Find the exact solution to $\sqrt{x-3} - \sqrt{x+1} = -2$.

Solution: Add $\sqrt{x+1}$ to both sides before squaring each side.

$$\sqrt{x-3} - \sqrt{x+1} + \sqrt{x+1} = -2 + \sqrt{x+1} \quad \text{addition property}$$
$$\sqrt{x-3} = \sqrt{x+1} - 2 \quad \text{simplification}$$
$$(\sqrt{x-3})^2 = (\sqrt{x+1} - 2)^2 \quad \text{power property}$$

Remember $(a+b)^2 = a^2 + 2ab + b^2$

$$x-3 = x+1-4\sqrt{x+1}+4$$
$$x-3 = x+5-4\sqrt{x+1} \quad \text{subtract } x \text{ and } 5$$
$$-8 = -4\sqrt{x+1} \quad \text{divide by } -4$$
$$2 = \sqrt{x+1} \quad \text{square both sides}$$
$$2^2 = (\sqrt{x+1})^2 \quad \text{simplify}$$
$$4 = x+1$$
$$x = 3$$

check for extraneous solution

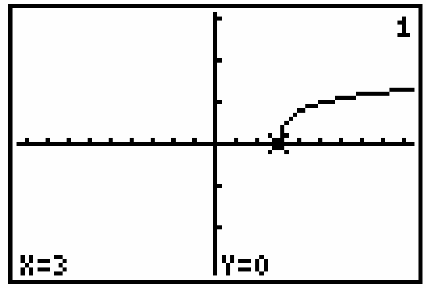
$$\sqrt{x-3} - \sqrt{x+1} = -2$$
$$\sqrt{3-3} - \sqrt{3+1} = -2$$
$$\sqrt{0} - \sqrt{4} = -2$$
$$-2 = -2$$

The solution is 3. Remember: A graphical check may be a better choice.

Example 9: Solve $\sqrt{x-3} - \sqrt{x+1} = -2$ again. **Continued**

</td></tr>
</table>

Solution: Since the exact solution is not required, you may want to use a graphical method. It too may give the exact solution however. Add 2 to both sides and graph the related function. Graph $y = \sqrt{x-3} - \sqrt{x+1} + 2$ and find values for *x* where the function has a value of 0.



The solution is exactly 3.

Remember, you have five methods for solving an equation. In addition to the three graphical methods and the analytical method, you may also use the numeric method.

Below is the numerical method for solving the above equation.

Table 9.5.1

x	3	4	5	6	7	8
$y = \sqrt{x-3} - \sqrt{x+1} + 2$	0	0.76	0.96	1.09	1.17	1.24

The domain is [3, ∞), thus there are no zeros left of 3. The function is increasing; therefore, there are no zeros to the right of 3. The only solution is 3.

- **Mathematical concepts are introduced in the context of real-world situations.**

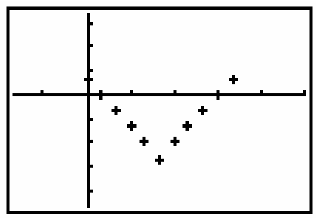
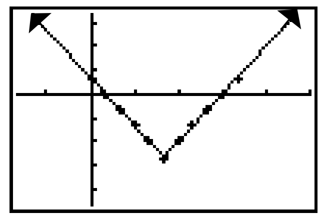
About 70% of the topics developed start with a problem that promotes the mathematics. Here is a sample taken from Section 3.3.

The Alberta Clipper

When a fast moving "Alberta Clipper" cold front drops down from Alberta, Canada and sweeps across a relatively small path through the Northeast quarter of the US, the temperature drops quickly by as much as 25° and then recovers quickly. Below is the numeric representation of the temperature (*T*) in Columbus, Ohio during an "Alberta Clipper". The day the front arrived, the daily high had been 18° at 4 PM. Time (*t*) zero is midnight.

t	0	1	2	3	4	5	6	7	8	9	10
T	2	0	-2	-4	-6	-8	-6	-4	-2	0	2

If a meteorologist wants to model this data with a function, what function can be used? The numeric data suggests that the linear function cannot be used because the temperature (the function) is both decreasing and increasing -- not a behavior of the linear function. Can the data be modeled by a quadratic function? Quadratic functions both decrease and increase on the normal domain. However, the quadratic function may not be a good choice because the average rate of change of the quadratic is not constant. If you check the rate of change of the function in the numeric representation, you will find that from midnight to 5 am the temperature decreased by 2° per hour and after 5 am it increased by 2° per hour. This is not one of the behaviors of the quadratic function. If not linear and if not quadratic, then what? The graphical representation of the data may give a clue **Continued**

	as to the type of function that can be used to model the temperature. The shape matches that of the absolute value function studied in Sections1.2 and 2.2. The figure shows the data and the graph of the absolute value function $2\|t-5\|-8$ on the same coordinate system. The absolute value function $2\|t-5\|-8$ matches the data. The shape is a V and the graph of any absolute value function of the form $d\|x+e\|+f$ will always have this shape. You may want to confirm this conjecture by graphing several absolute value functions of the form $2\|t-5\|-8$. When an absolute value function looks like $d\|x+e\|+f$, it is in **standard form** and the function parameters are *d*, *e*, and *f*.  
• **Thorough analysis of applications.**	Many application situations have extended exercise questions. The above example has one related exercise in the exercise set—it is listed below. 7. The temperature in the "Alberta Clipper Problem" was 18° when it started and the temperature dropped to −8° 13 hours later, this is a drop of −26°. If a similar front moves through with the 4 PM beginning temperature of 33°, the model for the temperature becomes $2\|t-5\|+7$. For this clipper, find the minimum temperature, when the temperature reaches the minimum, when the temperature is dropping (decreasing), when the temperature is rising (increasing), how fast the temperature is decreasing and then increasing, when the temperature is zero, when it is negative, and when it is positive? Many sections have more. For example, Section 7.5 starts with the problem of mixing ground-up peanut shells with the cleaning agent in laundry detergent. The exercise set contains six related problems. This idea of using exercises related to problems developed in the section should make students realize that a "real-world" situation gives rise to multiple mathematical problems. Not unlike in the real world. This approach forces students to read the textbook in a new way. It becomes a reference book because they must refer to the initial situation for information to help solve exercise problems.
• **Guided discovery exercises and explorations**	Many sections contain discovery exercise sets and all sections contain exploration exercises. Below is a sample guided discovery exercise set from Section 9.1. Find the minimum or maximum value of the following functions; also, find the value of *x* at the minimum or maximum. 21. $f(x)=5.6\sqrt{x-3.2}+11.2$ 22. $y=-2\sqrt{x+7}+11.2$ **Continued**

	23. $f(x) = -3\sqrt{x-5} - 11.2$ 24. $y = 0.2\sqrt{x+4.4} - 11.2$ 25. What are the coordinates of the maximum (or minimum) point on the graph of $y = d\sqrt{x+e} + f$? What this and other guided exercises do is to make students think about what they have just learned in the exercises. Many times, students think of exercises as "just mindless practice." The fact is that students can learn from this kind of exercise. Upon successful completion of an exercise like this, students have learned mathematics on their own – one of the best learning experiences available. Below are two explorations from Section 4.5 where students have learned how to factor. 41. Pick numbers at random for the parameters *a*, *b*, and *c* in the trinomial of the form $ax^2 + bx + c$ and try to find the factors. How many are factorable? 43. Find the zeros of the following functions. a. $f(x) = (x-5)(x-1)(x+3)$ b. $f(x) = \|(x-5)(x-1)(x+3)\|$ c. $f(x) = \|(x-5)\|(x-1)(x+3)$ d. $f(x) = (x-5)\|(x-1)\|(x+3)$ e. $f(x) = (x-5)(x-1)\|(x+3)\|$ f. $f(x) = \|(x-5)(x-1)\|(x+3)$ g. $f(x) = (x-5)\|(x-1)(x+3)\|$ Explorations are usually extensions of material in the section. Most explorations in the text require a calculator. There is considerable variety in the types of explorations. In the second example above, students are expanding on the relationship between zeros and factors. They have never seen what these graphs look like, but they certainly should learn more about the connection between factors and zeros.
• **Reduced emphasis on the use of symbol manipulation and increased emphasis on the use of function as a central theme.**	While the number of pages and the number of exercises dedicated to the practice of symbol manipulation has been limited, this is not to say that these traditional topics have been eliminated. Traditional algorithms have a place in the reform curriculum. They are no longer "the" curriculum. What you won't see in this text is exercises or examples of simplification of expressions with three or four levels of exponents with three or four bases. You won't see many exercises with complicated radicals to **Continued…**

	be simplified. Nor are there sums or differences of fractions with denominators containing five or six linear factors. What is in the text is enough for courses that follow and enough for surviving in the world outside academia. One of the main features of this text, as recommended by current standards documents, is the use of function as a central theme. A traditional approach is to use equations as a central theme, but this has been a problem in mathematics education because it requires considerable symbol manipulation ability, if no technology is used. If technology alone is used to solve equations, there is little math to include in the text. However, the major reason for using a function approach is that: • Function behavior is much more prevalent in the world than are equations. (That is, we aren't just teaching engineers and scientists anymore.) • Equations are a subset of the study of functions. So the scientists and engineers still learn equation setup and solving. • All the traditional mathematics can be developed from the study of functions. • New topics like mathematical modeling and data analysis follow much more naturally. • Technology allows for the study of functions as they commonly appear – as data pairs and as graphs. If you think that the study of functions is too complicated for beginning and intermediate algebra students, you may be thinking of function notation. Even this can be developed smoothly because of technology. But, when function is used as a central theme, there are many opportunities to develop the concept throughout a course. Because functions permeate a course and are no longer in just one topic of a course, the exposure is much deeper and richer than when using an equation/manipulation approach.
• **Three methods (numeric, graphic, and algebraic) of representing a function.**	As students make the transition from number based thinking about mathematics to a symbolic-based approach it is highly desirable to keep referring to numeric representations. The first cycle of mathematics in the text explaining the behavior of functions is numeric. The second cycle is the use of graphical representations. The third is to use symbols. This is developed in Chapters Two and Three. From Chapter Two to the end, all three representations of relationships are used as appropriate for the situation. However, when analyzing a new idea, the text almost always starts with numeric information. This follows the pattern established in their life experiences with mathematics.
• **Distributed learning has been incorporated.**	For example, the idea of function is introduced in Section 2.0 as a data relationship. This intuitive idea of a function is further developed in Chapter Two. From the data relationships, algebraic expressions that model data relationships are developed in Section 2.2. Symbolic representation is then used in Chapter Three. Chapter Four continues with formal function notation, and functions are used in equation and inequality solving in Chapters Six, Seven, Eight, Nine, Ten, Thirteen and Fourteen. These chapters also offer analysis of individual function types. Each of these chapters starts with a relative through investigation of a particular function. The mathematical level of the work with functions in Chapter Two is relatively low. Chapter Three raises the level slightly, and the remaining chapters finish the distributed learning.
• **A variety of methods for solving problems are encouraged.**	The use of technology provides for a natural flow of using a variety of methods for solving problems as does using a function approach. Many of the worked-out sample problems show a variety of methods with a discussion as to why each is used.

Students are encouraged to explore on their own.	Below is an excerpt from Section 4.6 showing how students are encouraged to explore on their own. "This example is significant because the absolute value function has had its domain altered while the shape of the graph has remained the same. Hopefully, you have already come to expect that different symbolic representations of functions have different graphical representations. In the case of $\lvert x-3\rvert-4+0\sqrt{x}$, the graphical representation is still a **V** with a different domain. Explore a little with your calculator by entering any function of your choosing and then add the function $0\sqrt{x}$. Find the domain of your function. The beauty of adding $0\sqrt{x}$ is that you are adding 0 to the function. This is why the graph does not change, just the domain. Try graphing any function on your calculator. Now add $0\sqrt{x}$ to it and graph it again. Note that there is no change in shape, only a change in the domain. See the Explorations exercises for a further investigation of this idea."
• **Higher level thinking skills are encouraged through projects, open-ended questions, and concept questions.**	Every section contains open-ended questions and concept-questions. Most chapters contain extended projects. Below are typical exercises from Section 5.3 that are based on concepts rather than skills. 36. What information about the graphical representation of a linear function is needed to find the symbolic representation? 37. If the only thing known about the graph of a linear function is its slope, can one distinct symbolic representation be found? Explain your answer. 38. Can more than one symbolic representation be found for a line through (2, 1)? Explain your answer. 39. Can a single symbolic representation be found for a line whose *y*-intercept is 6? Explain. 40. If the graph of $y = ex + f$ is perpendicular to $y = gx + h$, what is the relationship between e and g? 41. If the graph of $y = ex + f$ is parallel to $y = gx + h$, what is the relationship between e and g? Are f and h related? 42. Make a list of applications of the linear function. Include background material and uses of the linear function. 43. What is the meaning of the words "point-slope form of a linear function?" 44. What conditions must be met before two lines can be perpendicular? 45. Describe how you put the function $y=-2(x+4)-5$ in slope-intercept form. Do not actually put it in slope-intercept form. 46. Describe how you can put $y = 2x + 5$ in point-slope form. Do not actually do it. Below are examples of open-ended questions found in typical exercise sets. These are taken from Section 3.4. 22. Create a square root function that is increasing. 23. Create a square root function that is decreasing. 24. Create a square root function that has a maximum of 7. 25. Create a square root function that has a minimum of 2. 26. Create a square root function that has a domain of $[-3, \infty)$. 27. Create a square root function that has a range of $[-3, \infty)$. 28. Create a square root function that has a zero at 4. 29. Give an example of a square root function whose graph crosses the *x*-axis. 30. Give an example of a square root function whose graph does not cross the *x*-axis. 31. Give an example of a square root function that starts on the *x*-axis.

<table>
<tr>
<td>• Sample problems are checked numerically or graphically and students are encouraged to do likewise.</td>
<td>Many times they are checked algebraically because they are solved by technology-based methods. Below is an example from Section 9.5.

Example 7: Find the exact solution to $x-3=\sqrt{2x+2}$.
Solution:
$x-3=\sqrt{2x+2}$ given
$(x-3)^2=(\sqrt{2x+2})^2$ power property of equality
$x^2-6x+9=2x+2$ simplification
$x^2-8x+7=0$ subtraction property of equality
$(x-1)(x-7)=0$ factorization
$x=1$ $x=7$ relationship between factors and zeros
Check for extraneous solutions:
$x-3=\sqrt{2x+2}$ $x-3=\sqrt{2x+2}$
$1-3=\sqrt{2\cdot 1+2}$ $7-3=\sqrt{2\cdot 7+2}$
$-2=\sqrt{4}$ $4=\sqrt{16}$
$-2=2$ FALSE $4=4$ TRUE
$x\neq 1$ $x=7$ This is the only solution.

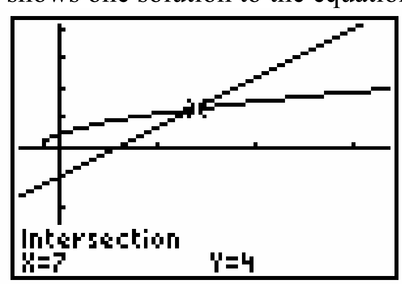
Instead of checking analytically, you may find it simpler to check graphically. Figure 8.5.9 shows the graphs of the functions $x-3$ and $\sqrt{2x+2}$. As you can see the intersection method only shows one solution to the equation.


Figure 9.5.9</td>
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<td>• Exercise sets contain low-level difficulty problems that prime students for future topics.</td>
<td>To cause students to think about future topics and to instill a sense of comfort with the topics, each exercise set starts with a series of four (usually simple) questions that foreshadow what is to come. One use of the questions is for discussion at the end (beginning) of each class period. Below is an example from Section 7.1.

-4. Simplify $(x\cdot x\cdot x\cdot x)\ (x\cdot x\cdot x\cdot x)\ (x\cdot x\cdot x\cdot x)\ (x\cdot x\cdot x\cdot x)\ (x\cdot x\cdot x\cdot x)$
-3. Is $6=2^{x+1}-4$ an equation or a function?
-2. Give an example of something that grows exponentially.
-1. What is the strength of a solution that is 12% alcohol?

Question number -4 is to prepare students for the next section that is on the properties of exponents. Question -3 is a reminder of the difference between equations and functions. Topic 7.3 is on solving exponential equations. Question –2 is to start students thinking about applications of the exponential function studied in 7.4. Finally, question -1 is to prime students for the rational function developed in the first section of the next chapter.</td>
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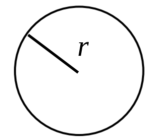
• **Exercise sets contain high-level difficulty problems that review topics previously taught.**	These six exercises found in every section are usually designed to enhance skills from the previous four sections. It is a simple device that helps students remember skills longer through distributive learning. Below are the exercises from Section 10.1, which is on the quadratic function. 1. The radius of a circle can be found by the function $r = \frac{\sqrt{A}}{\sqrt{\pi}}$. This simplifies to $r = 0.5642\sqrt{A}$. Is the function increasing? What is the domain of the function? Find A when $r = 10$ inches.  2. Imagine a coiled spring hanging from the ceiling, with a mass attached to the spring. If the mass is pulled down a small amount and released, it will bounce up and down and have a period of $T = \frac{2\pi}{\sqrt{k}}\sqrt{m}$. The period is T seconds for a mass of m. The value of k is the spring constant and is different for each spring. 3. Suppose the spring constant of a certain spring is 0.01. The function then simplifies to $T = 20\pi\sqrt{m}$, or simply $T = 62.83\sqrt{m}$. If a mass of 7 grams is set in motion, what is its period? What is the period for a 14-gram mass? 3. In the above situation, what mass (in grams) can be hung from the spring to maintain a period of 2 seconds? 4. Solve $-\sqrt{2x-6}+3=0$ 5. Solve $\sqrt{x+4}+\sqrt{4-x}=3$ 6. Simplify $\left[\left(256^{\frac{1}{2}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ These six exercises are set at the beginning of each exercise set to emphasize the importance of reviewing previous material. Three exercises are from the previous section, two are from the second previous section and one is from the third previous section.
• **Exercise sets contain writing questions. The more modes of teaching we can use, the more likely our students will <u>correctly</u> remember what we teach.**	Every exercise set has writing questions. They can be assigned as part of a regular assignment. They may be used as journal entries. While some writing questions, like those below, are standard to every section. Each section also has a unique set of questions. 60. After reading this section, make a list of questions that you want to ask your instructor. 61. Continue in your daily journal and make an entry. In addition to your normal entry on thoughts about the mathematics in this section, list at least two positive comments about what you have learned about this topic. 62. In paragraph format, summarize the material in this section of the text in your daily journal. **Continued…**

	63. Describe how your classroom instructor made this topic more understandable and clear. 64. After reading the text and listening to your instructor, what do you not understand about this topic? Below are writing questions specific to Section 14.1. 56. Explain why the base of the logarithmic function cannot be a negative number. 57. In the function in Exercise 53, can $(x+e)$ be a negative number? Why? 58. In the function in Exercise 53, can *y* be a negative number? Why? 59. What happens when you try to find log (-5) on your calculator? Explain why you think the calculator does what it does.
• **Exercise sets contain exploration problems. These are in activity format in the ancillary book.**	Many of the explorations can be used for group work. Many can be used as portfolio exercises. Below are samples from Section 5.4. Please note that students have not learned about transformation in Chapter Five. 25. From Exercise 14 - the waiter problem - graph the salary model followed by a model that shows a wage of \$90 plus tips. Of \$100 plus tips. Of \$110 plus tips. How are all of the graphs related? From Exercise 14 - the waiter problem - graph the salary model followed by a model that shows a wage of \$80 plus tips at a rate of \$3.75 per hour. At \$4.00 per hour. At \$4.25 per hour. At \$4.50 per hour. How are the graphs related? 26. From Exercise 12 - the gas-tank problem – graph the gas model followed by a model that shows a tank size of 13 gallon. Of 13.5 gallons. Of 16 gallons. How are all of the graphs related? From Exercise 12 - the gas tank problem – graph the gas model followed by a model that shows a rate of 35 mpg. Of 37 mpg. Of 40 mpg. Of 43 mpg. How are the graphs related? What is the problem domain for each of these models?
• **Exercise sets contain open-ended questions. These are in activity format in the ancillary book.**	Every exercise set contains open-ended questions. They are quite challenging to many students. You may use the questions as take-home quizzes. Below are a few examples from Section 6.3. 30. In the inequality $\|x + 3\| +$ **?** < 0, what value can the "**?**" be replaced with so the inequality has no solution? (Hint: There are many answers.) 31. In the inequality $-2\|x + 4\| +$ **?** > 0, describe all possible numbers that "**?**" can be replaced with so there is no solution. 32. In the inequality $-2\|x + 4\|$ **?** 0, what inequality symbol can "**?**" be replaced with so the inequality has no solution? 33. Create an equation containing an absolute function that has a solution of -3. 34. Find an equation containing an absolute function that has a solution of -4 and -2. 35. Develop an inequality containing an absolute function that has a solution of $[-4, -2]$. 36. Make an inequality containing an absolute function that has a solution of $(-\infty, -4] \cup [-2, \infty)$. 37. Write any inequality containing an absolute function that has no solution. 38. Create any inequality containing an absolute function that has $(-\infty, \infty)$ as a solution.

• **Extended laboratory projects on modeling are included with the text. These are in activity format in the ancillary book.**	The modeling projects have been tested over a period of six years and have proved to be extremely valuable, both as an alternative assessment tool and as a means of getting students to talk about mathematics. They have been used by a variety of people at several colleges. The modeling projects provide you with an opportunity to ask your students to apply the mathematics that is in the text. Mathematics such as using the relationships between the parameters in a function and the behavior of the function, geometric transformations, arithmetic operations of functions and the resulting change in the domain, and behavior near the zeros of functions. Students must recognize the shapes of the basic elementary functions. They must know how arithmetic operations of functions changes the geometry of the graphical representation of the functions. Your students must use three different representations of functional relationships found in the data. Each project contains Either the numeric representation or the graphical representation of a functional relationship. The main goal of the modeling projects is to find a good symbolic representation of any function that models the data. Just as an engineer, physicist, business person, or astronomer may solve a problem by following a prescribed procedure, your students are directed through this problem solving process by being asked a series of questions about the data. Once they have solved the main problem and have developed the symbolic representation of the function (mathematical model), they are asked to defend their solution by explaining the limitations of their model. This includes giving a situation when their model does not apply. They must explain their thinking on how they developed the model. Students are given the opportunity to use their model when asked questions about data not available in the given information. They are also asked to conjecture on what type of professional person might use their model. And finally, they must identify the references and resource person(s) used to help them in the problem solving process. Class time may be used to work the projects or they provide a good assignment outside of class. Depending on the level of mathematical sophistication of your students, the projects may take from three to fifteen or so hours. Group work is encouraged. As you move away from evaluating your students by traditional testing methods, the modeling projects should provide a large percent of your assessment of student understanding of mathematics. It is very useful to discuss various student answers during class time. Students also should be given the answers you develop.

e-Activities for *Foundations* (directions for teachers)

The Texas Instruments (TI) StudyCard app (free calculator software) and companion StudyCard Creator (free computer software) help create and facilitate the e-activities described below. The StudyCard stacks are called e-activities because they are all electronically based. Meaning that no hard copies are needed. They are provided to you and your students in electronic form, they are executed in electronic form on the calculator, you can edit my work with StudyCard Creator- electronically, and you distribute them to your students electronically via the calculator-to-calculator link cable, graph link cable from your computer, or wirelessly via TI Navigator™.

The e-activities have three uses, as a teaching tool – by the teacher as a lesson plan and used during class (like a Power Point presentation), by the student before attending class, and the other use is as a summative assessment. If used by the student, either outside class or during class, it is recommended that students work in a team of two. In addition to the benefits of collaborative work, students also have access to the functionality of the TI-83/84 Plus (or SE). The purpose and type of each activity is marked in the table below. Those marked as summative assessments are used after a particular topic is developed in class and/or through a teaching activity. Those marked as teaching activities are used to help you teach a topic. Use as a teaching tool is described below.

Imagine asking your students a question. You might have 3 – 4 students raise their hands to answer and you will ask one student to respond. So, what you know for sure is that one student knows the answer to your question. But, what about everyone else? Can you assume they all know the answer? Even after the one student tells the answer, do the others understand? We often incorrectly assume that they do. When you use the e-activities, every student must answer all of the questions. Further, they can answer the questions in the e-activities in the privacy of their two-person group. That in its self is a powerful teaching method.

The teaching e-activities consist of a series of questions about a particular topic. They are arranged as in guided discovery learning, or as you might structure a lesson plan. But they are more than just a series of multiple-choice questions. In addition to the responses to the question being carefully selected, the backside of each study card (where you go, after a response is selected) usually contains an explanation of the correct response, or why incorrect responses are incorrect. Sometimes the backside of the card provides information that leads to the next question in the activity, or guides the student in another direction. On occasion, the backside asks them if they used the calculator for help in answering the question. While the e-activity is scored (inside the calculator), the intent of the teaching activities is learning, not assessment. However, since the score is recorded, you may use it as part of your overall assessment if you choose.

Teaching through asking a series of questions is a well-established and successful teaching method. If you choose to use the e-activities as your lesson plan, they will guide your in-class presentation of a mathematical topic. Because you project what is on your calculator screen to the full class, the topic will be at the center of the discussion, just like when you use a Power Point presentation. You may choose, or require your students to follow along with the e-activity on their calculators, so they are engaged too and not just watching your screen. Of course, you may put a question on the screen and ask them to answer it on their calculators. If you use TI Navigator, you would know how every student answered – immediately. If you do not use TI Navigator, you may ask for a show of hands, before moving on to the next question. The direction you take in class depends on how all students respond. After displaying the backside of the card, you may want, or need, to expand on the information before going on to the next question.

If you assign the teaching activities to be executed right before your next class starts, your students will be ready for your classroom lesson plan. This approach is suggested so as to not use valuable class time. On the other hand, the logistics may be too complicated. If this is the case, then you may choose to give your students 10 minutes at the beginning of class to do an activity – perhaps while you take attendance or other housekeeping tasks. The e-activities are not meant to replace the print copy explorations, concept quizzes, investigations, or modeling projects, but rather, to supplement the over-all teaching tools that are significant part of *Foundations for College Mathematics 2e*.

Due to the limitations of the app, proper mathematical notation cannot always be used on the calculator. For example, square root symbols, the symbol for infinity, two-line fractions, etc. Please have your students write e-activity questions with pencil and paper if the notation in the activity is too complicated. Notation used: sqr for square root, inf for infinity, / for division, * for multiplication, belongs to for $\in$, ^ for exponentiation, and log2(x) for $\log_2 x$.

Research in learning with hand-held technology is clear and consistent, if you want the full benefit of the technology, students must have access to the technology both in class and out. Research is also clear and consistent on increased test scores for those using the technology. The point is, that we have provided you with the activities (both the e-activities and the print activities) to assist you in teaching algebra for understanding, but to take advantage of these, students must have access to the technology at all times. Teaching from *Foundations* and its ancillaries means you have a wide variety of educational tools with which to engage students and teach for mathematical understanding. In addition to the text, which was written for students, you also have all the student activities from *Explorations, Concept Quizzes, Investigations, Writing Mathematics, and Modeling Projects for Foundations for College Mathematics.* You now also have 48 StudyCard activities that can be used in three ways.

The study card files are called AppVar files (on your calculator). The free study card app (StudyCrd.8Xk) from Texas Instruments is used to process the AppVars on the calculator. In particular, these AppVars are StudyCard stacks, and the app is the StudyCrd software. These can be used on any TI-83 Plus, TI-83 Plus SE, TI-84 Plus, TI-84 Plus SE calculator. They can be copied to your calculator through TI-Connect. Texas Instruments TI-Connect is free computer software that allows your PC to communicate with your calculator through the USB GraphLink cable (gray or black cables will work too).

All of the stacks are editable through the use of the free computer software StudyCard Creator. It can be found on the Texas Instruments web page.

To execute a stack on the TI-83 Plus (or other), from the app key, select the StudyCard app and then follow the on-screen directions to open a stack. The front or back of each card may often be more than one calculator screen. If it is, the down arrow cursor movement key on the calculator can be used to see the remaining card. Flipping the card from front to back will be considered as missing the question. Cards can be re-played using the left arrow cursor movement key.

Below is a list of the stacks and below that a discussion of how to use them.

Computer file name	Calculator file name (what your students see)	Chapter-Section	Suggestions on when to use	Description
BEHAVABS.8xv	Behaviors-AbsValue	Chapter Three, Section Three; & perhaps Chapter Six, Section Three	As a lead-in to 3.3. As a review before you teach 6.3. If you teach the parameter-behavior connection, use it then too.	A teaching activity for the absolute value function that includes an absolute value data relationship, symbolic form, and absolute value behaviors such as opening up or down, increasing/decreasing, slope of each branch, location of the vertex, maximum, minimum, and range. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the absolute value function.
BEHAVEXP.8xv	Behaviors-Exp	Chapter Seven, Section One	As a lead-in to 7.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity for the exponential function that includes exponential data relationships, symbolic form, and exponential behaviors such as increasing/decreasing, horizontal asymptotes, and range. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the exponential function.
BEHAVLIN.8xv	Behaviors-Linear	Chapter Three, Section One; Chapter Five, Section One	As a lead-in to 3.1 or 5.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity for the linear function that includes linear data relationships, symbolic form, and linear behaviors such as increasing/decreasing, introduction to rate of change, *y*-intercept, and zeros. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the linear function.
BEHAVRTN.8xv	Behaviors-Rational	Chapter Eight, Section One	As a lead-in to 8.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity for rational functions that includes a rational data relationship, symbolic form, and rational behaviors such as increasing/decreasing, domain, range, horizontal and vertical asymptotes, and zeros. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the rational function.
BEHAVSQR.8xv	Behaviors-SquareRt	Chapter Three, Section Four;	As a lead-in to 3.4 or 9.1 –	A teaching activity for square root functions that includes a square root

		Chapter Nine, Section One	either to students right before (or at the beginning of) class, or as your presentation.	data relationship, symbolic form, and square root behaviors such as increasing/decreasing, domain, range, and maximum/minimums. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the square root function.
BEHVQUAD.8xv	Behaviors-Quadrati	Chapter Three, Section Two; Chapter Ten, Section One	As a lead-in to 3.2 or 10.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity for quadratic functions that includes quadratic data relationships, symbolic form, and quadratic behaviors such as opening up or down, increasing/decreasing, location of the vertex, maximum, minimum, and range. The emphasis is on making the connection between the function parameters and the resulting geometric behaviors of the quadratic function.
CHANGE.8xv	Change	Chapter Two, Section Three; Chapter Five, Section One	As a lead-in to 2.3 or 5.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity that demonstrates that the constant rate of change idea is present in many situations outside the mathematics classroom.
CREATABS.8xv	CreateAbsoluteValu	Chapter Three, Section Three; & maybe Chapter SIx, Section Three	This is a summative assessment and should be used at the end of 3.3 and maybe the end of 6.3.	A summative assessment activity that tests students on the behaviors of the absolute value function. Students are also asked to create their own absolute function that meets the criteria listed in each card. Behaviors addressed are opening up or down, increasing/decreasing, slope of each branch, location of the vertex, maximum, minimum, and range. The emphasis is on assessing knowledge of the connection between the function parameters and the resulting geometric behaviors of the absolute value function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
CREATEXP.8xv	Create Exponential	Chapter Seven	Use at the end of Chapter Seen.	A summative assessment activity that tests students on the behaviors of the exponential function. Students are also asked to create their own exponential function that meets the criteria listed in each card. Behaviors addressed are increasing/decreasing, horizontal

				asymptotes, and range. The emphasis is on assessing knowledge of the connection between the function parameters and the resulting geometric behaviors of the exponential function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
CREATLIN.8xv	Create Linear	Chapter Five	Use at the end of Chapter Five.	A summative assessment activity that tests students on the behaviors of the linear function. Students are also asked to create their own linear function that meets the criteria listed in each card. Behaviors addressed are increasing/decreasing, rate of change, and *y*-intercept. The activity emphasis is on assessing knowledge of the connection between the function parameters and the resulting geometric behaviors of the linear function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
CREATQDR.8xv	Create Quadratic	Chapter Ten or maybe after Chapter Three, Section Two	Use at the end of Chapter Ten.	A summative assessment activity that tests students on the behaviors of the quadratic function. Students are also asked to create their own quadratic function that meets the criteria listed in each card. Behaviors addressed are increasing/decreasing, location of the vertex, maximum, minimum, range, positive/negative, and zeros. The activity emphasis is on assessing knowledge of the connection between the function parameters and the resulting geometric behaviors of the quadratic function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
CREATRTN.8xv	Create Rational	Chapter Eight	Use at the end of Chapter Eight.	A summative assessment activity that tests students on the behaviors of the rational function. Students are also asked to create their own rational function that meets the criteria listed in each card. Behaviors addressed are increasing/decreasing, domain, range, and horizontal and vertical asymptotes. The activity emphasis is on assessing

				knowledge of the connection between the function parameters and the resulting geometric behaviors of the rational function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
CREATSQR.8xv	CreateSquareRoot	Chapter Nine or maybe Chapter Three, Section Four	Use at the end of Chapter Nine, but also maybe at the end of 3.4.	A summative assessment activity that tests students on the behaviors of the square root function. Students are also asked to create their own square root function that meets the criteria listed in each card. Behaviors addressed are increasing/decreasing, domain, range, and maximum/minimums. The activity emphasis is on assessing knowledge of the connection between the function parameters and the resulting geometric behaviors of the square root function. Students are also encouraged to use the graphing calculator to help answer the questions as needed.
DOMRANGE.8xv	Domain & Range	Chapter Two, Section Two; Chapter Three, Section One	As a lead-in to 2.2 or 3.1 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity uses real-world contexts to teach the concepts of independent and dependent variables, and then domain and range. It includes a couple exercise-type examples at the end.
FACTOR1.8xv	Factoring-#1	Chapter Four, Section Five	Use it before you teach pencil and paper factoring!	A teaching activity that makes the equivalence and zeros connection between, for example, the function $y = (x - 3)(x + 2)$ and $y = x^2 - x - 6$. All quadratics included have a leading coefficient of 1.
FACTOR2.8xv	Factoring-#2	Chapter Four, Section Five, or maybe in section Four	Use it before you teach pencil and paper factoring!	A teaching activity that makes the equivalence and zeros connection between, for example, the function $y = (2x - 3)(5x + 2)$ and $y = 10x^2 - 11x - 6$. All quadratics included have a leading coefficient of something other than 1.
FACTOR3.8xv	Factoring-#3	Chapter Four, Section Five, or maybe in section Four	Use it before you teach pencil and paper factoring!	A teaching activity that makes the argument that if you know the zeros of, for example, $y = (2x - 1)(x + 3)$, (whether through the parameter-zero connection or using a graphing calculator), and you know the

				equivalence to $y = 2x^2 + 5x - 3$, then if you start with a function like $y = 2x^2 + 5x - 3$ and find the zeros using a graphing calculator, you know it is equivalent to $y = (2x - 1)(x + 3)$. Thus, students can now factor with the graph of the function.
FUNCTION.8xv	FunctionNotation	Chapter Four, Section One	As a lead-in to 4.1 – either to students right before (or at the beginning of) class, or as your lesson.	A teaching activity to help understand the meaning of the notation $f(x)$. In addition, cards also address finding, for example, $f(2)$ given $f(x)$, and the connection to the point on the graph of $f(x)$. The graphing calculator is used extensively.
INTERVAL.8xv	IntervalNotation	Chapter One, Section Three	As a lead-in to 1.3 – either to students right before (or at the beginning of) class, or as your lesson.	A teaching activity on understanding interval notation. It uses functions and function behaviors as the context for needing and using interval notation.
LAWSEXPO.8xv	LawsOfExponents	Chapter Seven, Section Two; or maybe parts in Chapter One, Section Four	As a lead-in to 7.2 – either to students right before (or at the beginning of) class, or as your presentation.	This activity is not on the laws of exponents, but rather, on the definitions of Natural number exponents, zero, and negative integer exponents. The activity uses student's understanding of behaviors of functions. In particular, students need to know the connection between the symbolic form and the graphical form of basic functions. The stack is instructive in nature. The title is used to associate it with the laws of exponents.
LAWSEXI.8xv	LawsOfExponentsI	Chapter Seven, Section Two; or maybe parts in Chapter One, Section Four	As a lead-in to 7.2 – either to students right before (or at the beginning of) class, or as your presentation.	This is a teaching activity on the First Law of Exponents and is based in the connection between functions in symbolic and graphic forms. This is to say that the first law is discovered by students through the use of multiplication of functions – comparing graphical representations with symbolic forms. Later confirmation of products is accomplished through comparing numeric representations of the problem and student answers.
LAWSEXII.8xv	LawsOfExponentsII	Chapter Seven, Section	As a lead-in to 7.2 – either to	This is a teaching activity on the Second Law of Exponents and is based

		Two	students right before (or at the beginning of) class, or as your presentation.	in the connection between functions in symbolic and graphic forms. This is to say that the second law is discovered by students through the use of division of functions – comparing graphical representations with symbolic forms.
LAWEXIII.8xv	LawsOfExponentsIII	Chapter Seven, Section Two	As a lead-in to 7.2 – either to students right before (or at the beginning of) class, or as your presentation.	This is a teaching activity on the Third Law of Exponents and is based in the connection between functions in symbolic and graphic forms. This is to say that the third law is discovered by students through the use of raising power functions to powers – comparing graphical representations with symbolic forms. Later confirmation of powers of powers is accomplished through comparing numeric representations of the problem and student answers.
MAXMINID.8xv	Max/Min & Inc/Dec	Chapter Two, Section Three, but it could be used in Chapter Three, 5.1, 7.1, 8.1, 9.1 and 10.1 as a review of the behaviors	As a lead-in to 2.3 – either to students right before (or at the beginning of) class, or as your presentation.	This is a teaching activity that uses real-world contexts to assist students in understanding the concepts of maximum, minimum, increasing, and decreasing.
POLYADD.8xv	PolynomialAddSubt	Chapter Four, Section Two	As a lead-in to 4.2 – either to students right before (or at the beginning of) class, or as your presentation.	Addition and subtraction of polynomials is based in the context of discovering that when linear functions are added, the slope of the sum is equal to the sum of the slopes, likewise for *y*-intercepts. Once students discover how to add (or subtract) linear polynomials (in the form of polynomial functions) the activity moves on to quadratic and up polynomials. Confirmation of sums or differences is accomplished through comparing numeric representations of the problem and the student answer.
POLYMULT.8xv	PolynomialMult	Chapter Four, Section Three	As a lead-in to 4.3 – either to students right before (or at the beginning of) class, or as your presentation.	Students discover that the product of linear functions (polynomial) is usually quadratic. They then move on to discovering the exact product. This leads to the more traditional methods for multiplying polynomials.

RATNLADD.8xv	RationalAddSubt	Chapter Eight, Section Four	As a lead-in to 8.4 – either to students right before (or at the beginning of) class, or as your presentation.	The activity is based in arithmetic to convince students they cannot add fractions by adding numerators and denominators. Further discovery focuses on adding (or subtracting) numerators with examples from simple to complex denominators. The teaching activity ends with sums or differences of rational functions with different denominators. Confirmation of sums or differences is accomplished through comparing numeric representations of the problem and student answers. A discussion of domain is included.
RATNLMUL.8xv	RationalMultiply	Chapter Eight, Section Three	As a lead-in to 8.3 – either to students right before (or at the beginning of) class, or as your presentation.	Rational expression multiplication and division algorithms are based in arithmetic, but then progress to function operations. Guided discovery concepts are used to develop ideas for the operations. Confirmation of products or quotients is accomplished through comparing numeric representations of the problem and student answers. A discussion of domain is included.
RATNLRED.8xv	RationalReduce	Chapter Right, Section Two	As a lead-in to 8.2 – either to students right before (or at the beginning of) class, or as your presentation.	This is a guided discovery teaching activity that starts with arithmetic examples and continues through algebra. The activity uses student's understanding of behaviors of functions. In particular, students need to know the connection between the symbolic form and the graphical form of basic functions. Confirmation of the correctness of the reduced form is accomplished through comparing numeric representations of the problem and student answers. A discussion of domain is included.
REALPROP.8xv	Real # Properties	Chapter One, Section One	As a lead-in to 1.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity on understanding the commutative properties, associative properties and the distributive property. Examples from arithmetic are used to lead to the properties.
SCIENTIF.8xv	ScientificNotation	Chapter One, Section Four	As a lead-in to 1.4 – either to students right before (or at the beginning of)	The case is built for the need of scientific notation by using very large numbers (and numbers near zero) from real-world contexts. Further, a guided discovery argument is use to get the

			class, or as your presentation.	students to the proper form. This is followed by examples.
SCIENTOP.8xv	SciTifOperations	Chapter One, Section Four	As a lead-in to 1.4 – either to students right before (or at the beginning of) class, or as your presentation.	The four basic operations are discussed in guided discovery fashion. Understanding is assessed by reversing the problem – given the answer to a (× or ÷), find two numbers.
SLOPE.8xv	Slope	Chapter Five, Section One	As a lead-in to 5.1 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity on understanding the meaning of slope as well as the evaluation of slope using $\Delta y/\Delta x$ definition. The activity starts with the function concept of increasing & decreasing and this leads to "how fast" is it increasing or decreasing. A connection is made to the linear function parameter that controls slope. It is somewhat more traditional in nature.
SOLVEABS.8xv	SolveAbsValueEqn	Chapter Six, Section Three	As a lead-in to 6.3 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity begins with "what is an equation?" and continues with a meaning of solving an equation. The initial method discussed is trace, as related to the absolute value function. That is, the function approach is used to solve the equations. An understanding of absolute value function behaviors is required.
SOLVEEXP.8xv	SolveExponentialEq	Chapter Seven, Section Three	As a lead-in to 7.3 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity begins with "what is an equation?" and continues with a meaning of solving an equation. The function approach is used to solve the equations and the exponential function behaviors are needed to help answer some questions.
SOLVELIN.8xv	SolveLinearEquatio	Chapter Six, Section One	As a lead-in to 6.1 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity begins with "what is an equation?" and continues with a meaning of solving an equation. A discovery approach is used to learn how to solve linear equations using the graph of the related linear function(s). The trace and zeros methods are developed – with emphasis on the connection between roots and zeros of the related function.
SOLVELOG.8xv	SolveLogEquations	Chapter Fourteen,	As a lead-in to 14.3 – either to	This teaching activity begins with "what is an equation?" and continues

		Section Three	students right before (or at the beginning of) class, or as your lesson.	with a meaning of solving an equation. The activity continues by developing the connection between points on the graph of the related function and a solution to an equation. Examples follow.
SOLVERTN.8xv	SolveRationalEqn	Chapter Eight, Section Five	As a lead-in to 8.5 – either to students right before (or at the beginning of) class, or as your lesson.	This teaching activity begins with “what is an equation?” and continues with a meaning of solving an equation. The activity continues by developing the connection between points on the graph of the related function and a solution to an equation. Examples follow.
SOLVESQR.8xv	SolveSqrRootEqn	Chapter Nine, Section Five	As a lead-in to 9.5 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity begins with “what is an equation?” and continues with a meaning of solving an equation. The activity continues by developing the connection between points on the graph of the related function and a solution to an equation. Examples follow, and extraneous roots are discussed.
SOLVINEQ.8xv	SolveInequalities	Chapter Six, Section Two	As a lead-in to 6.2 – either to students right before (or at the beginning of) class, or as your presentation.	The teaching activity starts with a discussion of the meaning of a solution to an inequality. A graphical method is assumed. The lesson ends with several examples.
SOLVQUAD.8xv	SolveQuadraticEqn	Chapter Ten, Section Two	As a lead-in to 10.2 – either to students right before (or at the beginning of) class, or as your presentation.	This teaching activity begins with “what is an equation?” and continues with a meaning of solving an equation. The activity focuses on “understanding” the solving process and the solution. Examples follow.
SQRADSUB.8xv	SqrRootAddSubtract	Chapter Nine, Section Three	As a lead-in to 9.3 – either to students right before (or at the beginning of) class, or as your presentation.	A direct connection is made from square root functions to adding and subtracting square root expressions. The teaching activity is based on the discovery approach and uses students’ knowledge of function behaviors. The activity ends with several examples.
SQRMULT.8xv	SqrMultiplication	Chapter Nine, Section Three	As a lead-in to 9.3 – either to students right	A brief discussion of the product property is followed by several “instructive” examples. Functions are

			before (or at the beginning of) class, or as your presentation.	used in early examples because students can use the calculator to confirm statements made.
SQRSIMP.8xv	SqrRootSimplifying	Chapter Nine, Section Two	As a lead-in to 9.2 – either to students right before (or at the beginning of) class, or as your presentation.	This is a teaching activity that starts by developing a need for simplification of square roots. Both numbers with perfect square factors and fractions are included in the sample exercises. It ends with finding square roots of numbers squared – leading to the square root of $\sqrt{x^2}$ being $\lvert x \rvert$.
VARDIRCT.8xv	Variation-Direct	Chapter Six, Section Four	As a lead-in to 6.4 – either to students right before (or at the beginning of) class, or as your presentation.	A teaching activity that connects the linear function $y = mx$ to the more traditional direct variation concepts. There are examples with explanations.
VARINVAR.8xv	Variation-Inverse	Chapter Eight, Section Five	As a lead-in to 8.5 – either to students right before (or at the beginning of) class, or as your presentation.	This is a teaching activity that uses real-world contexts to teach the idea of inverse variation. Inverse variation is then connected to the rational function and it ends with an example.
ZEROSPN.8xv	Zero's, Pos & Neg	Chapter Two, Section Three, or Chapter Three, Section One	As a lead-in to 2.3 – either to students right before (or at the beginning of) class, or as your presentation. May be reused in 3.1	In this teaching activity, several situations are included to teach the concepts of zeros, and when functions are positive or negative.