

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper

Exercise A, Question 1

Question:

Use the standard results for $\sum_{r=1}^n r$ and for $\sum_{r=1}^n r^2$ to show that, for all positive integers n , $\sum_{r=1}^n (r+1)(3r+2) = n(an^2 + bn + c)$, where the values of a , b and c should be stated.

Solution:

$$\begin{aligned} \sum_{r=1}^n (r+1)(3r+2) &= \sum_{r=1}^n (3r^2 + 5r + 2) \\ &= 3 \sum_{r=1}^n r^2 + 5 \sum_{r=1}^n r + 2 \sum_{r=1}^n 1 \\ &= 3 \frac{n}{6}(n+1)(2n+1) + 5 \frac{n}{2}(n+1) + 2n \\ &= \frac{n}{2}[(n+1)(2n+1) + 5(n+1) + 4] \\ &= \frac{n}{2}[2n^2 + 3n + 1 + 5n + 5 + 4] \\ &= \frac{n}{2}[2n^2 + 8n + 10] \\ &= n[n^2 + 4n + 5] \end{aligned}$$

So $a = 1$, $b = 4$ and $c = 5$.

Multiply out brackets first

Split into three separate parts to isolate $\sum r^2$, $\sum r$ and $\sum 1$

Use standard formulae for $\sum r^2$, $\sum r$ and remember that $\sum_{r=1}^n 1 = n$.

Take out factor $\frac{n}{2}$

Multiply out the terms in the bracket.

Simplify the bracket.

Take out factor of 2 from bracket which will then be 'cancelled' by the $\frac{1}{2}$ term to give the answer.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 2

Question:

$$f(x) = x^3 + 3x - 6$$

The equation $f(x) = 0$ has a root α in the interval $[1, 1.5]$.

a Taking 1.25 as a first approximation to α , apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation to α . Give your answer to three significant figures.

b Show that the answer which you obtained is an accurate estimate to three significant figures.

Solution:

a

$$f(x) = x^3 + 3x - 6$$

Differentiate $f(x)$ to give $f'(x)$

$$f'(x) = 3x^2 + 3$$

Using the Newton-Raphson procedure
with $x_1 = 1.25$

$$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)}$$

State the Newton-Raphson procedure.

$$= 1.25 - \frac{[1.25^3 + 3 \times 1.25 - 6]}{[3 \times 1.25^2 + 3]}$$

Substitute 1.25.

$$= 1.25 - \frac{[-0.296875]}{7.6875}$$

$$= 1.25 + .0386 \dots$$

$$= 1.29(\text{to } 3 \text{ sf})$$

Give your answer to the required accuracy.

b

$$f(1.285) = -0.023 \dots < 0$$

$$f(1.295) = 0.0567 \dots > 0$$

Check the sign of $f(x)$ for the lower and upper bounds of values which round to 1.29 (to 3 sf).

As there is a change of sign and $f(x)$ is continuous the root α satisfies

State 'sign change' and draw a conclusion.

$$1.285 < \alpha < 1.295$$

$\therefore \alpha = 1.29$ (correct to 3 sf).

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Edexcel AS and A Level Modular Mathematics

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Exercise A, Question 3

Question:

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ and } \mathbf{S} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

a Describe fully the geometric transformation represented by each of **R** and **S**.

b Calculate **RS**.

The unit square, U , is transformed by the transformation represented by **S** followed by the transformation represented by **R**.

c Find the area of the image of U after both transformations have taken place.

Solution:

a

R represents a rotation of 135° anti-clockwise about 0.

R takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ so is rotation.

S represents an enlargement scale factor $\sqrt{2}$ centre 0

S is of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ so is enlargement with scale factor k .

b

$$\mathbf{RS} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Use the process of matrix multiplication eg $(ab)\begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$.

c

Determinant of **RS** = 2

\therefore Area scale factor of U is 2.

\therefore Image of U has area 2.

Recall that the determinant of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$ and that this represents an area scale factor.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 4

Question:

$$f(z) = z^4 + 3z^2 - 6z + 10$$

Given that $1 + i$ is a complex root of $f(z) = 0$,

a state a second complex root of this equation.

b Use these two roots to find a quadratic factor of $f(z)$, with real coefficients.

Another quadratic factor of $f(z)$ is $z^2 + 2z + 5$.

c Find the remaining two roots of $f(z) = 0$.

Solution:

a

$1 - i$ is a second root.

This is the conjugate of $1 + i$, and complex roots of polynomial equations with real coefficients occur in conjugate pairs.

b

$[z - (1 + i)][z - (1 - i)]$ is a quadratic factor.

Multiply the two linear factors to give a quadratic factor.

$\therefore z^2 - 2z + 2$ is the factor.

c

$$\text{If } z^2 + 2z + 5 = 0$$

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4 - 20}}{2} \\ &= -1 \pm \frac{1}{2}\sqrt{16}i \\ &= -1 \pm 2i \end{aligned}$$

Use the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Remaining roots are $-1 + 2i$ and $-1 - 2i$.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 5

Question:

The rectangular hyperbola H has equation $xy = c^2$. The points $P \left(cp, \frac{c}{p} \right)$ and $Q \left(cq, \frac{c}{q} \right)$ lie on the hyperbola H .

a Show that the gradient of the chord PQ is $-\frac{1}{pq}$.

The point $R, \left(3c, \frac{c}{3} \right)$ also lies on H and PR is perpendicular to QR .

b Show that this implies that the gradient of the chord PQ is 9.

Solution:

a

The gradient of the chord PQ is $\frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$

$$= c \frac{(q-p)}{pq} \div c(p-q)$$

$$= c \frac{(q-p)}{pq} \times \frac{1}{c(p-q)}$$

$$= -\frac{(p-q)}{pq(p-q)}$$

$$= -\frac{1}{pq}$$

$$\text{Use gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

Use a common denominator to combine the fractions.

Express $(q-p)$ as $-(p-q)$

Divide numerator and denominator by the factor $(p-q)$.

b

PR has gradient $\frac{-1}{3p}$

QR has gradient $\frac{-1}{3q}$

These lines are perpendicular

$$\therefore \frac{-1}{3p} \times \frac{-1}{3q} = -1$$

$$\therefore \frac{1}{9pq} = -1$$

$$\therefore \frac{1}{pq} = -9$$

$$\therefore \text{Gradient of } PQ = \frac{-1}{pq} = 9.$$

Use the result established in part (a) to deduce these gradients.

Use the condition for perpendicular lines
 $mm' = -1$.

Find the value of $\frac{-1}{pq}$.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 6

Question:

$$\mathbf{M} = \begin{pmatrix} x & 2x-7 \\ -1 & x+4 \end{pmatrix}$$

a Find the inverse of matrix \mathbf{M} , in terms of x , given that \mathbf{M} is non-singular.

b Show that \mathbf{M} is a singular matrix for two values of x and state these values.

Solution:

a The determinant of \mathbf{M} is

$$\begin{aligned} & x(x+4) - (-1)(2x-7) \\ &= x^2 + 4x + 2x - 7 \\ &= x^2 + 6x - 7 \end{aligned}$$

The inverse of \mathbf{M} is

$$\frac{1}{x^2 + 6x - 7} \begin{pmatrix} x+4 & 7-2x \\ 1 & x \end{pmatrix}$$

b \mathbf{M} is singular when

$$\begin{aligned} & x^2 + 6x - 7 = 0 \\ \text{ie: } & (x+7)(x-1) = 0 \\ \therefore & x = -7 \text{ or } 1. \end{aligned}$$

Use the result that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Put the value of the determinant of \mathbf{M} equal to zero.

Then solve the quadratic equation.

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 7

Question:

The complex numbers z and w are given by $z = \frac{7-i}{1-i}$, and $w = iz$.

a Express z and w in the form $a + ib$, where a and b are real numbers.

b Find the argument of w in radians to two decimal places.

c Show z and w on an Argand diagram

d Find $|z - w|$.

Solution:

a

$$\begin{aligned} z &= \frac{7-i}{1-i} = \frac{(7-i)(1+i)}{(1-i)(1+i)} \\ &= \frac{8+6i}{2} \\ &= 4+3i \end{aligned}$$

Multiply numerator and denominator by the conjugate of $1 - i$.

Remember $i^2 = -1$

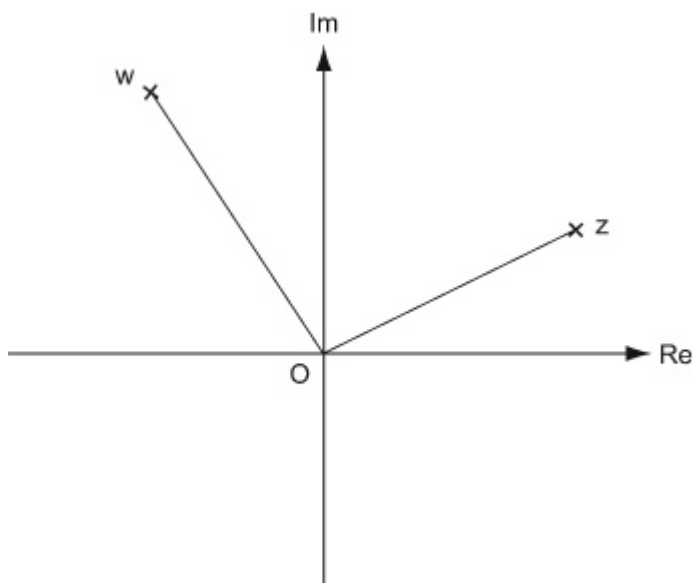
$$\begin{aligned} w = iz &= i(4+3i) \\ &= -3+4i \end{aligned}$$

b

$$\begin{aligned} \arg w &= \pi - (\tan^{-1} 4/3) \\ &= 2.21 \end{aligned}$$

As w is in the second quadrant in the Argand diagram.

c



d

$$\begin{aligned}z - w &= 7 - i \\|z - w| &= \sqrt{7^2 + (-1)^2} \\&= \sqrt{50} \\&= 5\sqrt{2}.\end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 8

Question:

The parabola C has equation $y^2 = 16x$.

a Find the equation of the normal to C at the point P , $(1, 4)$.

The normal at P meets the directrix to the parabola at the point Q .

b Find the coordinates of Q .

c Give the coordinates of the point R on the parabola, which is equidistant from Q and from the focus of C .

Solution:

a

$$y^2 = 16x \Rightarrow y = 4x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 4 \times \frac{1}{2} x^{-\frac{1}{2}}$$

$$= 2x^{-\frac{1}{2}}$$

At $(1, 4)$ gradient is 2

\therefore Gradient of normal is $\frac{-1}{2}$

The equation of the normal is $y - 4 = \frac{-1}{2}(x - 1)$

ie: $y = \frac{-1}{2}x + 4\frac{1}{2}$

b

The directrix has equation $x = -4$.

Substitute $x = -4$ into normal equation

$\therefore y = 6\frac{1}{2}$

So Q is the point $(-4, 6\frac{1}{2})$.

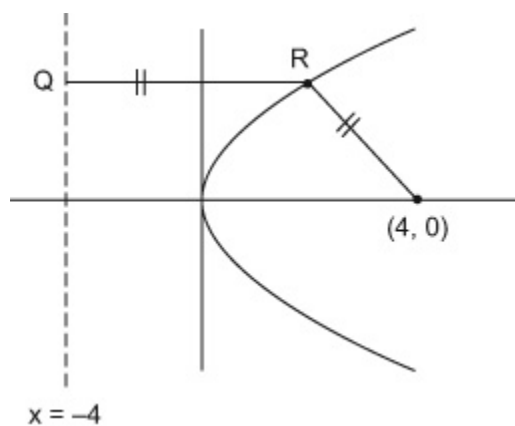
c

Find the gradient of the curve at $(1, 4)$.

Use $mm' = -1$ as the normal is perpendicular to the curve.

Use $y - y_1 = m(x - x_1)$

The directrix of the parabola $y^2 = 4ax$ has equation $x = -a$.



At R $y = 6\frac{1}{2}$

$$\therefore \left(6\frac{1}{2}\right)^2 = 16x$$

$$\therefore x = \frac{6\frac{1}{2} \times 6\frac{1}{2}}{16} = \frac{169}{64}$$

So R is the point $\left(\frac{169}{64}, \frac{13}{2}\right)$

The point R must have the same y -coordinate as the point Q .

Solutionbank FP1

Edexcel AS and A Level Modular Mathematics

Examination style paper

Exercise A, Question 9

Question:

a Use the method of mathematical induction to prove that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^n r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}(n^2 + n + 4) - \left(\frac{1}{2}\right)^{n-1}.$$

b $f(n) = 3^{n+2} + (-1)^n 2^n, n \in \mathbb{Z}^+$.

By considering $2f(n+1) - f(n)$ and using the method of mathematical induction prove that, for $n \in \mathbb{Z}^+$, $3^{n+2} + (-1)^n 2^n$ is divisible by 5.

Solution:

a Let $n = 1$

$$LHS = 1 + \left(\frac{1}{2}\right)^0 = 1 + 1 = 2$$

$$\begin{aligned} RHS &= \frac{1}{2}(1^2 + 1 + 4) - \left(\frac{1}{2}\right)^0 \\ &= \frac{1}{2} \times 6 - 1 = 2 \end{aligned}$$

Show that the result is true when $n = 1$.

$\therefore LHS = RHS$ so result is true for $n = 1$

Assume that the result is true for $n = k$

$$\text{ie: } \sum_{r=1}^k \left[r + \left(\frac{1}{2}\right)^{r-1} \right] = \frac{1}{2}(k^2 + k + 4) - \left(\frac{1}{2}\right)^{k-1}$$

Add $(k+1) + \left(\frac{1}{2}\right)^k$ to each side.

Show that assuming the result is true for $n = k$ implies that it is also true for $n = k + 1$

$$\begin{aligned} \therefore \sum_{r=1}^{k+1} r + \left(\frac{1}{2}\right)^{r-1} &= \frac{1}{2}(k^2 + k + 4) + (k+1) - \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k \\ &= \frac{1}{2}(k^2 + k + 4 + 2k + 2) + \left(\frac{1}{2}\right)^{k-1} \left(-1 + \frac{1}{2}\right) \quad \text{Collect the similar terms together.} \\ &= \frac{1}{2}(k^2 + 3k + 6) - \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} \\ &= \frac{1}{2}((k+1)^2 + (k+1) + 4) - \left(\frac{1}{2}\right)^k \end{aligned}$$

$$\text{ie: } \sum_{r=1}^n r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}(n^2 + n + 4) - \left(\frac{1}{2}\right)^{n-1}$$

where $n = k + 1$

ie: Result is implied for $n = k + 1$.

\therefore By induction, as result is true for $n = 1$ then it is implied for $n = 2, n = 3, \text{etc.}$ ie: for all positive integer values for n .

Conclude that this implies by induction that the result is true for all positive integers.

b

$$f(n) = 3^{n+2} + (-1)^n 2^n \quad n \in \mathbb{Z}^+$$

Let $n = 1$

$$\begin{aligned} f(1) &= 3^3 + (-1)^1 2^1 \\ &= 27 - 2 \\ &= 25 \end{aligned}$$

Show that the result is true when $n = 1$.

This is divisible by 5.

Let $f(k)$ be divisible by 5

Assume that $f(k)$ is divisible by 5

$$\text{ie: } 3^{k+2} + (-1)^k 2^k = 5A \quad *$$

Consider

$$\begin{aligned} 2f(k+1) - f(k) &= 2 \cdot 3^{k+3} + 2(-1)^{k+1} 2^{k+1} - 3^{k+2} - (-1)^k 2^k \\ &= 3^{k+2} [2 \cdot 3 - 1] + 2^k (-1)^k [-4 - 1] \\ &= 3^{k+2} \times 5 - 5 \cdot (-1)^k 2^k \\ &= 5(3^{k+2} - (-1)^k 2^k). \end{aligned}$$

Follow the hint given in the question

Collect similar terms together and look for common factor of 5.

$\therefore 2f(k+1) - f(k)$ is divisible by 5.

$= 5B$

$$\begin{aligned} \therefore 2f(k+1) &= 5B + f(k) \\ &= 5(B + a) \end{aligned}$$

As $f(k)$ and $2f(k+1) - f(k)$ are each divisible by 5, deduce that $f(k+1)$ is also divisible by 5.

ie: $2f(k+1)$ is divisible by 5 $\Rightarrow f(k+1)$ is divisible by 5.

So by induction as $f(1)$ is divisible by 5 then so is $f(2)$ and so is $f(3) \dots$ and by induction $f(n)$ is divisible by 5 for all positive integers n .

Use induction to complete your proof.