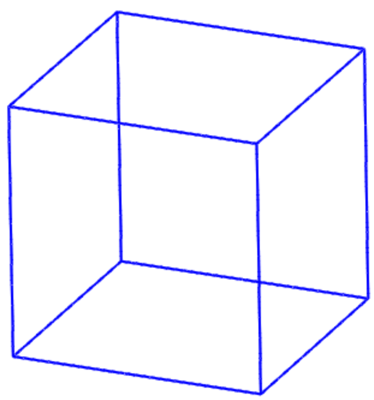
Calculus 3 - Divergence Theorem

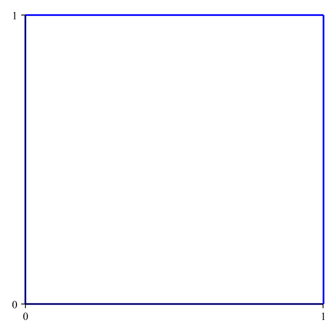
Flux

Last class we introduced flux

$$\iint_S \vec{F} \cdot \vec{N} dS. \tag{1}$$

and a particular problem was to find the flux across the unit cube $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $0 \leq z \leq 1$ where $\vec{F} = \langle x, xy, xyz \rangle$





Soln: Since there are 6 sides to the cube we must do all 6 fluxes separately. The nice thing is that the unit normal's are easy to pick off and so are dS.

Top: Here $\vec{N} = \langle 0, 0, 1 \rangle$. Since $z = 1$, then $\vec{F} = \langle x, xy, xy \rangle$ and $\vec{F} \cdot \vec{N} = \langle 0, 0, 1 \rangle \cdot \langle x, xy, xy \rangle = xy$

$$\iint_S \vec{F} \cdot \vec{N} dS = \int_0^1 \int_0^1 xy dy dx = \frac{1}{4} \tag{2}$$

Bottom: Here $\overrightarrow{N} = \langle 0, 0, -1 \rangle$. Since $z = 0$, then $\overrightarrow{F} = \left\langle x, xy, 0 \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 0, 0, -1 \rangle \cdot \left\langle x, xy, 0 \right\rangle = 0$

$$\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 0 dy dx = 0 \tag{3}$$

Right: Here $\overrightarrow{N} = \langle 0, 1, 0 \rangle$. Since $y = 1$, then $\overrightarrow{F} = \left\langle x, x, xz \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 0, 1, 0 \rangle \cdot \left\langle x, x, xz \right\rangle = xy$

$$\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 x dz dx = \frac{1}{2} \tag{4}$$

Left: Here $\overrightarrow{N} = \langle 0, -1, 0 \rangle$. Since $y = 0$, then $\overrightarrow{F} = \left\langle x, 0, 0 \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 0, -1, 0 \rangle \cdot \left\langle x, 0, 0 \right\rangle = 0$

$$\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 0 dz dx = 0 \tag{5}$$

Front: Here $\overrightarrow{N} = \langle 1, 0, 0 \rangle$. Since $x = 1$, then $\overrightarrow{F} = \left\langle 1, y, yz \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle 1, 0, 0 \rangle \cdot \left\langle 1, y, yz \right\rangle = 1$

$$\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 1 dz dy = 1 \tag{6}$$

Back: Here $\overrightarrow{N} = \langle -1, 0, 0 \rangle$. Since $x = 0$, then $\overrightarrow{F} = \left\langle 0, 0, 0 \right\rangle$ and $\overrightarrow{F} \cdot \overrightarrow{N} = \langle -1, 0, 0 \rangle \cdot \left\langle 0, 0, 0 \right\rangle = 0$

$$\iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \int_0^1 \int_0^1 0 dz dy = 0 \tag{7}$$

$$\text{Total Flux} \quad \iint\limits_S \overrightarrow{F} \cdot \overrightarrow{N} dS = \frac{1}{4} + 0 + \frac{1}{2} + 0 + 1 + 0 = \frac{7}{4}. \tag{8}$$

Divergence Theorem

Let V be a solid region bound by a closed surface S oriented by a outward unit normal $\vec{N}$. If $\vec{F}$ is a vector field whose components have continuous first derivatives in V then

$$\iint\limits_S \vec{F} \cdot \vec{N} dS = \iiint\limits_V \nabla \cdot \vec{F} dV \tag{9}$$

Example 1.

We consider the problem stated above. We first calculate the divergence of $\vec{F}$ so

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(xyz) = 1 + x + xy \tag{10}$$

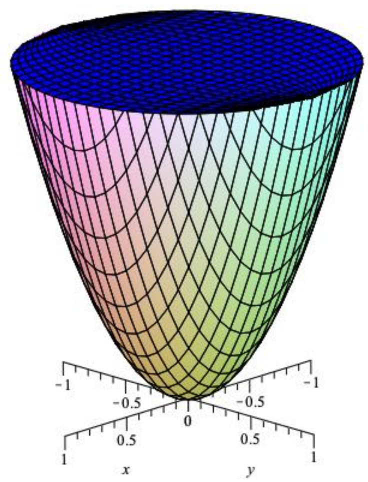
We integrate this over the volume of the cube so

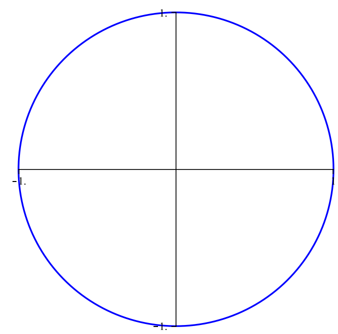
$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 (1 + x + xy) dz dy dx &= \int_0^1 \int_0^1 (1 + x + xy) z \Big|_0^1 dy dx \\ &= \int_0^1 \int_0^1 (1 + x + xy) dy dx \\ &= \int_0^1 (y + xy + \frac{1}{2} x y^2) \Big|_0^1 dx \\ &= \int_0^1 (1 + \frac{3}{2} x) dx \\ &= (x + \frac{3}{4} x^2) \Big|_0^1 \\ &= \frac{7}{4} \end{aligned} \tag{11}$$

which is the same answer found on the previous page.

Example 2. Verify the Divergence theorem where the vector field is $\vec{F} = \langle xz, yz, 1\rangle$ and the volume bound by the surfaces $z = x^2 + y^2$ and $z = 1$.

Soln. We first calculate the flux integrals. As there are two surfaces there





will be two fluxes.

Top. Here the normal is $\vec{N} = \langle 0, 0, 1\rangle$ and on this surface ($z = 1$) $\vec{F} = \langle x, y, 1\rangle$ so $\vec{F} \cdot \vec{N} = 1$ and the flux is

$$\iint_S \vec{F} \cdot \vec{N} dS = \iint_R 1 dA = 1 \tag{12}$$

Side. If we define $G = x^2 + y^2 - z$ then the normal is

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \frac{\langle 2x, 2y, -1\rangle}{\sqrt{4x^2 + 4y^2 + 1}} \tag{13}$$

and it is outward! Next we calculate dS which is give by

$$dS = \sqrt{1 + f_x^2 + f_y^2} dA_{xy} = \sqrt{1 + 4x^2 + 4y^2} dA_{xy} \tag{14}$$

thus, the flux out of the paraboloid is

$$\begin{aligned}\iint_S \vec{F}\cdot\vec{N}dS &= \iint_S \langle xz,\, yz,\, 1\rangle \cdot \frac{\langle 2x,\, 2y,\, -1\rangle}{\sqrt{4x^2+4y^2+1}}\sqrt{1+4x^2+4y^2}dA_{xy} \\ &= \iint_S \left(2x^2z + 2y^2z - 1\right) dA_{xy} \quad \text{(bring in surface)} \\ &= \iint_{R_{xy}} \left(2(x^2+y^2)^2 - 1\right) dA_{xy} \quad \text{(switch to polar)} \\ &\int_0^{2\pi}\int_0^1 (2r^4-1)rdrd\theta = -\frac{\pi}{3}\end{aligned} \tag{15}$$

so the total flux is

$$\iint_S \vec{F}\cdot\vec{N}dS = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \tag{16}$$

For the second part, we calculate the divergence of $\vec{F}$ so

$$\nabla\cdot\vec{F} = z + z = 2z.$$

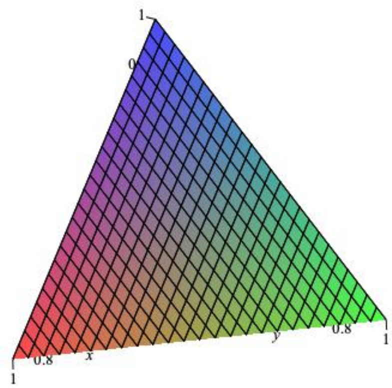
The volume integral is

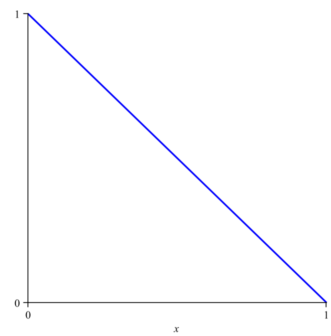
$$\begin{aligned}\iiint_V 2zdV &= \int_0^{2\pi}\int_0^1\int_{r^2}^1 2zrdz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 rz^2\Big|_{r^2}^1 dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r(1-r^4)dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2}r^2 - \frac{1}{6}r^6\right)\Big|_0^1 d\theta \\ &= \int_0^{2\pi}\frac{1}{3}d\theta = \frac{2\pi}{3}\end{aligned} \tag{17}$$

verifying the Divergence theorem.

Example 3. Verify the Divergence theorem where the vector field is $\vec{F} = \left\langle 2xy, -y^2, z^2 \right\rangle$ and the volume bound by the surfaces $x + y + z = 1$ and the xy, xz and yz planes.

Soln. We first calculate the flux integrals. As there are four surfaces there





will be four fluxes.

Bottom: Here $\vec{N} = \langle 0, 0, -1 \rangle$. Since $z = 0$, then $\vec{F} = \left\langle 2xy, -y^2, 0 \right\rangle$ and $\vec{F} \cdot \vec{N} = \langle 0, 0, -1 \rangle \cdot \left\langle 2xy, -y^2, 0 \right\rangle = 0$ so no flux.

Left: Here $\vec{N} = \langle 0, -1, 0 \rangle$. Since $y = 0$, then $\vec{F} = \left\langle 0, 0, z^2 \right\rangle$ and $\vec{F} \cdot \vec{N} = \langle 0, -1, 0 \rangle \cdot \left\langle 0, 0, z^2 \right\rangle = 0$ so no flux.

Back: Here $\vec{N} = \langle -1, 0, 0 \rangle$. Since $x = 0$, then $\vec{F} = \left\langle 0, -y^2, z^2 \right\rangle$ and $\vec{F} \cdot \vec{N} = \langle -1, 0, 0 \rangle \cdot \left\langle 0, -y^2, z^2 \right\rangle = 0$ so no flux.

Plane: Since the surface is given as $x + y + z = 1$ we create G as $G = x + y + z - 1$. So $\nabla G = \left\langle 1, 1, 1 \right\rangle$ and the unit normal is given by

$$\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \frac{\left\langle 1, 1, 1 \right\rangle}{\sqrt{3}}. \tag{18}$$

Next, we calculate dS. Since the surface is given by $z = 1 - x - y$ then

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA_{xy} = \sqrt{1+1+1}\, dA_{xy}. \tag{19}$$

Now the flux integral becomes

$$\begin{aligned}
\iint\limits_S \vec{F} \cdot \vec{N} dS &= \iint\limits_S \left\langle 2xy,\, -y^2,\, z^2 \right\rangle \cdot \frac{\langle 1,1,1 \rangle}{\sqrt{3}} \sqrt{3}\, dA_{xy} \\
&= \iint\limits_S \left(2xy - y^2 + z^2\right) dA_{xy} \\
&= \int_0^1 \int_0^{1-x} \left(2xy - y^2 + (1-x-y)^2\right) dy\, dx \\
&= \frac{1}{12}
\end{aligned} \tag{20}$$

The total flux out of the tetrahedren is $\frac{1}{12}$. For the second part, we calculate the divergence of $\vec{F}$ so $\nabla \cdot \vec{F} = 2y - 2y + 2z = 2z$. The volume integral is

$$\begin{aligned}
\iiint\limits_V 2z dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2z dz\, dy\, dx \\
&= \int_0^1 \int_0^{1-x} z^2 \Big|_0^{1-x-y} dy\, dx \\
&= \int_0^1 \int_0^{1-x} (1-x-y)^2 dy\, dx \\
&= \frac{1}{12}
\end{aligned} \tag{21}$$

verifying the Divergence theorem.