Calculus 3 - Limits

In Calculus 1 we considered limits. For example

$$\lim_{x \to 1} \frac{x^2 + 2x - 2}{2x + 1}.$$

If we directly substitute $x = 1$ we get

$$\lim_{x \to 1} \frac{x^2 + 2x - 2}{2x + 1} = \frac{1^2 + 2 \cdot 1 - 2}{2 \cdot 1 + 1} = \frac{1}{3}$$

a simple number. However, if we consider

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{\text{“}0\text{”}}{\text{”}0\text{”}}, \tag{1}$$

with $\frac{\text{“}0\text{”}}{\text{”}0\text{”}}$ meaning nothing. So we consider alternate approaches to derive the limit. For example, graphically we see that (1) and from the graph we

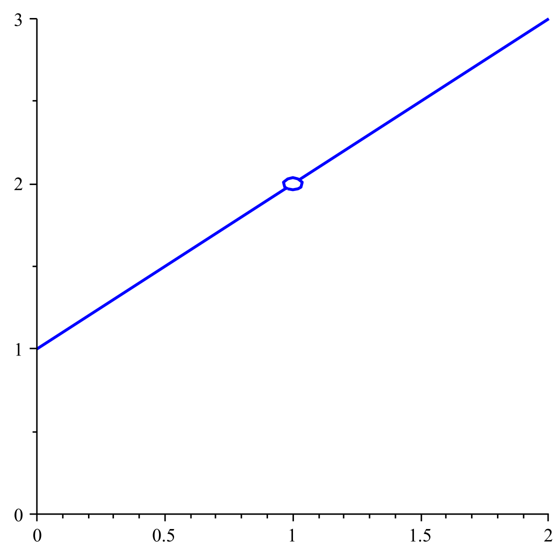

Figure 1: Graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$

determined that

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2. \tag{2}$$

Analytically, we we see

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} x + 1 = 2 \tag{3}$$

It's important to realize that not all limits exist. For example consider

$$f(x) = \begin{cases} 1 & \text{if } \; x < 0 \\ x & \text{if } \; x \geq 0 \end{cases} \tag{4}$$

and the limit

$$\lim_{x \to 0} f(x) \tag{5}$$

Clearly approaching zero from the left and from the right gives different

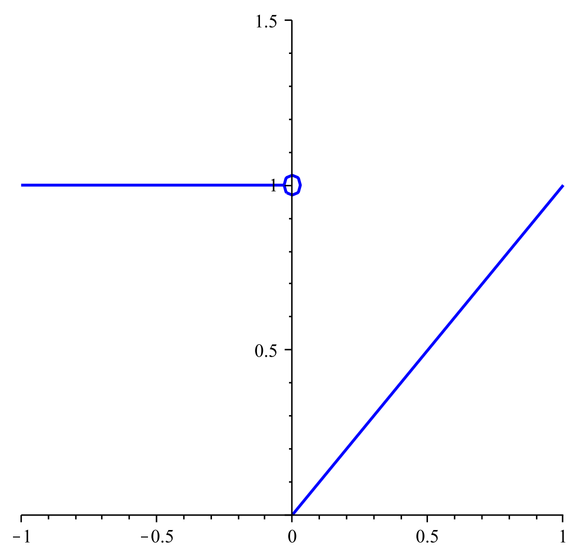

Figure 2: Branch function (4)

values. So we created one-sided limits

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0} 1 = 1 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0} x = 0 \end{aligned} \tag{6}$$

and since

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x) \tag{7}$$

then the limit does not exist (DNE).

In general we considered

$$\lim_{x\to a} f(x) = L \tag{8}$$

and a very formal way (using $\delta - \epsilon$) of proving that limits exist.

So now we extend limits to 3D and consider

$$\lim_{(x,y)\to(a,b)} f(x,y) \tag{9}$$

and ask - do these limits exist?

Consider for example,

$$\lim_{(x,y)\to(1,-1)} \frac{10xy - 2y^2}{x^2 + y^2} \tag{10}$$

Well, as a first approach, let's try a direct substitution. Doing so yields

$$\lim_{(x,y)\to(1,-1)} \frac{10xy - 2y^2}{x^2 + y^2} = \frac{-10 - 2}{1 + 1} = \frac{-12}{2} = -6 \tag{11}$$

and so the limit is equal to 6.

Consider

$$\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x} \tag{12}$$

a direct substitution yields

$$\lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x} = \frac{4-4}{-2+2} = \frac{\text{“}0\text{”}}{\text{”}0\text{”}} \tag{13}$$

so there's that $\frac{\text{“}0\text{”}}{\text{”}0\text{”}}$ again meaning we need to do something else. One will notice that factoring will work here since

$$\begin{aligned} \lim_{(x,y)\to(1,-2)} \frac{y^2+2xy}{y+2x} &= \lim_{(x,y)\to(1,-2)} \frac{y(y+2x)}{y+2x} \\ &= \lim_{(x,y)\to(1,-2)} y \\ &= -2 \end{aligned} \tag{14}$$

so the limit exists!

Consider

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{\text{“}0\text{”}}{\text{”}0\text{”}} \tag{15}$$

so we need to do something else. How many ways can we approach $(0,0)$? Well, there really are an infinite number of ways. For example, we could let $y=0$ and then let $x\to 0$ so we would be approaching $(0,0)$ along the x-axis. We could let $x=0$ and then let $y\to 0$ so we would be approaching $(0,0)$ along the y-axis. We could also let $y=x$ and then let $x\to 0$ so we would be approaching $(0,0)$ along the line $y=x$.

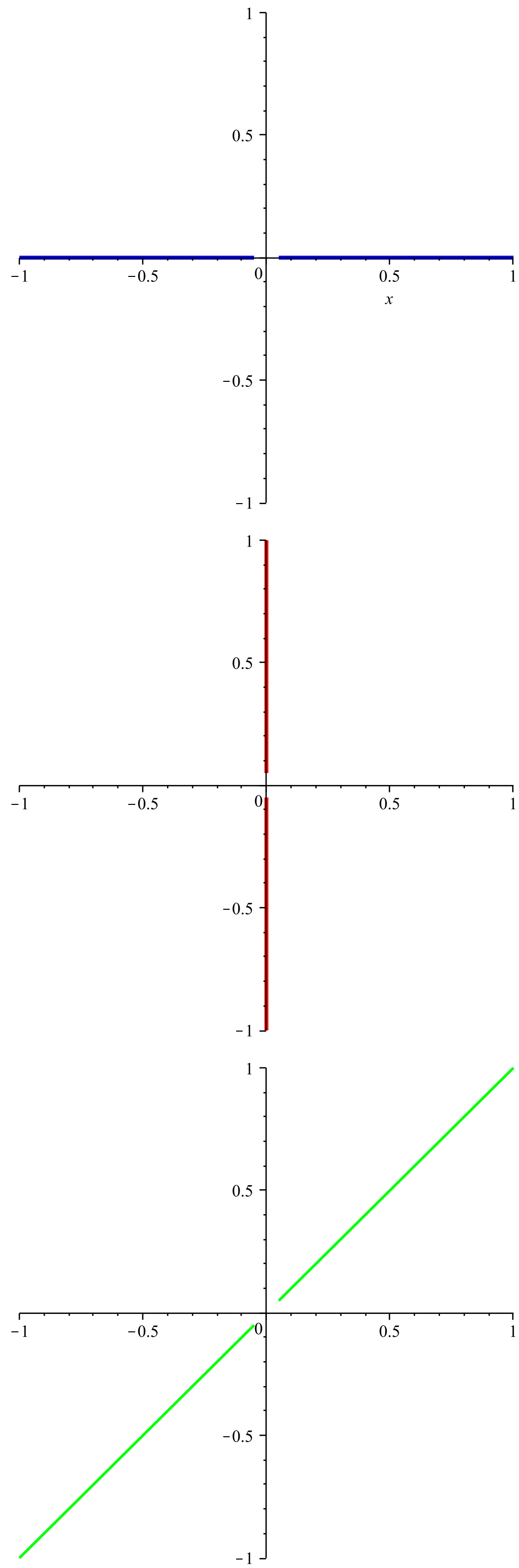

Figure 3: Following different paths

So let's see what happens in our limit (eqn. (15)) following the x and y axes.

$$\begin{aligned} &\text{along x axis } (y=0) \quad \lim_{(x,y)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1 \\ &\text{along y axis } (x=0) \quad \lim_{(x,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1 \end{aligned} \tag{16}$$

and since following different paths, we get different limits, the limit DNE!

Example 4

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{\text{``}0\text{''}}{\text{''}0\text{''}} \tag{17}$$

Along $x=0$ (y axis) and $y=0$ (x axis) we obtain

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0 \\ \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} &= \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0 \end{aligned} \tag{18}$$

and so you might be tempted to say the limit is 0 but if we follow $y=x$ then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0 \tag{19}$$

and so in this example, the limit DNE!

Example 5

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \frac{\text{``}0\text{''}}{\text{''}0\text{''}} \tag{20}$$

Along $x = 0$, $y = 0$ and $y = x$ we get

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} &= 0 \\ \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} &= 0 \\ \lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} &= \lim_{(x,y)\to(0,0)} \frac{2x^3}{x^4+x^2} = 0 \end{aligned} \tag{21}$$

and so you might be tempted to say the limit is 0 but if we follow $y = x^2$ then

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^4}{x^4+x^4} = \frac{2}{2} = 1 \neq 0 \tag{22}$$

and so in this example, the limit DNE!

Example 6

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \frac{\text{“}0\text{''}}{\text{''}0\text{''}} \tag{23}$$

Along $x = 0$, $y = 0$ and $y = x$ we get the limit is zero. So maybe the limit is actually zero. This we consider in the next class.