Math 1497 — Calc 2

8.3 Infinite Series

we now add up the terms in the sequence

so if $\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots \}$

then $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots$

if $a_n = \frac{1}{2^n}$

the series is $\sum_{n=1}^{\infty} \frac{1}{2^n}$

we've seen these before - Riemann Sums

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Here the upper limit of the index $\to \infty$

so we have an infinite series

We now ask Given a series

$$\sum_{n=1}^{\infty} a_n \quad \text{does it converge}$$

meaning does the sum $\to$ #

Before trying to answer this question, we will consider some special series

(1) Geometric Series

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

To get an ~~answ~~ answer to whether this converges we will add term by term

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

$S_5 = \frac{31}{32}$, $S_6 = \frac{63}{64}$ & so on it appears that

$$S_N = \frac{2^N - 1}{2^N} = 1 - \frac{1}{2^N}$$

and $\lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - \frac{1}{2^N} = 1$

The reason why it's called a geometric series is b/c the same number is used to multiply each term in series to get the next #

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \qquad \frac{1}{4} * \frac{1}{2} = \frac{1}{8} \qquad \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} \text{ etc}$$

Here the "common ratio" is $\frac{1}{2}$

To show the series conv. we will stop the series after N terms

$$S_N = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^{N-1}} + \frac{1}{2^N}$$

$$\frac{1}{2} S_N = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \frac{1}{2^N} - \frac{1}{2^{N+1}}$$

sub, $S_N - \frac{1}{2} S_N = \frac{1}{2} - \frac{1}{2^{N+1}} \rightarrow \frac{S_N}{2} = \frac{1}{2} - \frac{1}{2^{N+1}}$

and we get our formula again

$$S_N = 1 - \frac{1}{2^N}$$

In general, a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots$$

a – 1st term, r – common ratio

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \underline{\text{or}} \quad \sum_{n=0}^{\infty} ar^{n}$$

just changed the index

so after N

$$S_N = a + ar + ar^2 + ar^3 + \cdots ar^{N-1}$$

$$r S_N = ar + ar^2 + ar^3 + \cdots ar^{N-1} + ar^N$$

Sub

$$S_N - r S_N = a - ar^N$$

$$(1-r)\, S_N = \frac{a - ar^N}{\cancel{1-r}}$$

$$S_N = \frac{a}{1-r} - \frac{ar^N}{1-r}$$

<u>Note</u> if $r = 1$

$S = a + a + a + \cdots$

$\to \infty$

if $a \neq 0$

$a = 0$ gives $S \equiv 0$ always

$$\lim_{N\to\infty} S_N = \frac{a}{1-r} - \frac{a}{1-r}\lim_{N\to\infty} r^N$$

say $r=2$ $\quad \lim_{N\to\infty} 2^N \to \infty$

$r=\frac{1}{3}$ $\quad \lim_{N\to\infty} \left(\frac{1}{3}\right)^N \to 0$

so if $0<r<1$ then $\lim_{N\to\infty} S_N = \frac{a}{1-r}$

if $r\geq 1$ then $\lim_{N\to\infty} S_N \to \infty$

the series converges if $0<r<1$ $\quad S_\infty = \frac{a}{1-r}$ the actual sum

if $r\geq 1$ the series div

we can go further and say

if $-1<r<1$ $\quad \sum_{n=1}^{\infty} ar^n$ conv to $S_\infty = \frac{a}{1-r}$

if not it diverges

<u>Note</u> $-1<r<1 \iff |r|<1$

Ex 1 $\sum_{n=1}^{\infty} 7\left(\frac{1}{4}\right)^n = \frac{7}{4} + 7\left(\frac{1}{4}\right)^2 + 7\left(\frac{1}{4}\right)^3 + \cdots$

common ratio $r = \frac{1}{4}$ & yes $|r| < 1$

so this series conv. $S_\infty = \dfrac{\frac{7}{4}}{1-\frac{1}{4}} = \dfrac{7/4}{3/4} = 7/3$

Ex 2 $\sum_{n=3}^{\infty} +5(-5/4)^n = 5(-5/4)^3 + 5(-5/4)^4 + \cdots$

$r = -5/4$ and $|r| > 1$ so $\int$ diverges

Special Series #2 Telescopic

Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

why does this help. Stop at N & write out terms

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

drop ( ) & cancel

$$1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cdots \cancel{\tfrac{1}{N-1}} - \cancel{\tfrac{1}{N}} - \cancel{\tfrac{1}{N}} - \frac{1}{N+1}$$

$$S_N = 1 - \frac{1}{N+1}$$

Now let $N \to \infty$

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} 1 - \frac{1}{N+1} \to 1$$

the tail end of the remaining terms will determine what will happen

In general

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - b_{n+1} \qquad a_n \text{ splits}$$

(1) Stop at N
(2) write out terms
(3) cancel
(4) $N \to \infty$

Ex

$$\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} \ln n - \ln n+1$$

$$S_N = (\ln 1 - \cancel{\ln 2}) + (\cancel{\ln 2} - \cancel{\ln 3}) + \ldots (\ln N-1 - \ln N) + \ln N - \ln N+1$$

$$= \ln 1 - \ln N+1$$

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} - \ln N+1 \to -\infty \quad \text{so diverges}$$

Special Series #3 Harmonic

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

this series diverges. we group term.

$$S_1 = 1$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \overbrace{\frac{1}{4} + \frac{1}{4}}^{=1/2}$$

$$1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{3}{2}$$

$$S_{16} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \Big(\underbrace{\frac{1}{16} + \cdots \frac{1}{16}}_{8 \text{ of these}}\Big) = 1 + \frac{4}{2}$$

$S_{32} > 1 + \frac{5}{2}$

$S_{64} > 1 + \frac{6}{2}$

$\vdots$

$S_{2^n} > 1 + \frac{n}{2}$ as $n \to \infty$ $S_{2^n} > \infty$

$\therefore \lim_{n \to \infty} 1 + \frac{n}{2} \to \infty$

so the series diverge (ever so slowly)

#4 p series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$p > 1$ series converges

$p \leq 1$ series diverge

we'll show this tomorrow.