

Solutionbank S3

Edexcel AS and A Level Modular Mathematics

Combinations of random variables

Exercise A, Question 1

Question:

Given the random variables $X \sim N(80, 3^2)$ and $Y \sim N(50, 2^2)$ where X and Y are independent find the distribution of W where:

- a $W = X + Y$,
- b $W = X - Y$.

Solution:

$$\begin{aligned} \text{a} \quad E(W) &= E(X) + E(Y) \\ &= 80 + 50 \\ &= 130 \end{aligned}$$

$$\begin{aligned} \text{Var}(W) &= \text{Var}(X) + \text{Var}(Y) \\ &= 9 + 4 \\ &= 13 \\ W &\sim N(130, 13) \end{aligned}$$

$$\begin{aligned} \text{b} \quad E(W) &= E(X) - E(Y) \\ &= 80 - 50 \\ &= 30 \end{aligned}$$

$$\begin{aligned} \text{Var}(W) &= \text{Var}(X) + \text{Var}(Y) \\ &= 9 + 4 \\ &= 13 \\ W &\sim N(30, 13) \end{aligned}$$

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Exercise A, Question 2

Question:

Given the random variables $X \sim N(45, 6)$, $Y \sim N(54, 4)$ and $W \sim N(49, 8)$ where X , Y and W are independent, find the distribution of R where $R = X + Y + W$.

Solution:

$$\begin{aligned} E(R) &= E(X) + E(Y) + E(W) \\ &= 45 + 54 + 49 \\ &= 148 \end{aligned}$$

$$\begin{aligned} \text{Var}(R) &= \text{Var}(X) + \text{Var}(Y) + \text{Var}(W) \\ &= 6 + 4 + 8 \\ &= 18 \end{aligned}$$

$$R \sim N(148, 18)$$

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Exercise A, Question 3

Question:

X_1 and X_2 are independent normal random variables. $X_1 \sim N(60, 25)$ and $X_2 \sim N(50, 16)$. Find the distribution of T where:

- a $T = 3X_1$,
- b $T = 7X_2$,
- c $T = 3X_1 + 7X_2$,
- d $T = X_1 - 2X_2$.

Solution:

- a** $E(T) = 3E(X_1)$
 $= 3 \times 60$
 $= 180$
 $\text{Var}(T) = 9 \text{Var}(X_1)$
 $= 9 \times 25$
 $= 225$
 $T \sim N(180, 225)$ or $N(180, 15^2)$
- b** $E(T) = 7E(X_2)$
 $= 7 \times 50$
 $= 350$
 $\text{Var}(T) = 49 \text{Var}(X_2)$
 $= 49 \times 16$
 $= 784$
 $T \sim N(350, 784)$ or $N(350, 28^2)$
- c** $E(T) = E(3X_1) + E(7X_2)$
 $= 180 + 350$
 $= 530$
 $\text{Var}(T) = \text{Var}(3X_1) + \text{Var}(7X_2)$
 $= 225 + 784$
 $= 1009$
 $T \sim N(530, 1009)$
- d** $E(T) = E(X_1) - 2E(X_2)$
 $= 60 - 2 \times 50$
 $= -40$
 $\text{Var}(T) = \text{Var}(X_1) + 4\text{Var}(X_2)$
 $= 25 + 4 \times 16$
 $= 89$
 $T \sim N(-40, 89)$

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Exercise A, Question 4

Question:

Y_1, Y_2 and Y_3 are independent normal random variables.

$Y_1 \sim N(8, 2), Y_2 \sim N(12, 3)$ and $Y_3 \sim N(15, 4)$. Find the distribution of A where:

a $A = Y_1 + Y_2 + Y_3,$

b $A = Y_3 - Y_1,$

c $A = Y_1 - Y_2 + 3Y_3,$

d $A = 3Y_1 + 4Y_3,$

e $A = 2Y_1 - Y_2 + Y_3.$

Solution:

$$\begin{aligned}
 \mathbf{a} \quad E(A) &= E(Y_1) + E(Y_2) + E(Y_3) \\
 &= 8 + 12 + 15 \\
 &= 35
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(A) &= \text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) \\
 &= 2 + 3 + 4 \\
 &= 9
 \end{aligned}$$

$$A \sim N(35, 9) \text{ or } N(35, 3^2)$$

$$\begin{aligned}
 \mathbf{b} \quad E(A) &= E(Y_3) - E(Y_1) \\
 &= 15 - 8 \\
 &= 7
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(A) &= \text{Var}(Y_3) + \text{Var}(Y_1) \\
 &= 4 + 2 \\
 &= 6
 \end{aligned}$$

$$A \sim N(7, 6)$$

$$\begin{aligned}
 \mathbf{c} \quad E(A) &= E(Y_1) - E(Y_2) + 3E(Y_3) \\
 &= 8 - 12 + 3 \times 15 \\
 &= 41
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(A) &= \text{Var}(Y_1) + \text{Var}(Y_2) + 9\text{Var}(Y_3) \\
 &= 2 + 3 + 9 \times 4 \\
 &= 41
 \end{aligned}$$

$$A \sim N(41, 41)$$

$$\begin{aligned}
 \mathbf{d} \quad E(A) &= 3E(Y_1) + 4E(Y_3) \\
 &= 3 \times 8 + 4 \times 15 \\
 &= 84
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(A) &= 9\text{Var}(Y_1) + 16\text{Var}(Y_3) \\
 &= 9 \times 2 + 16 \times 4 \\
 &= 82
 \end{aligned}$$

$$A \sim N(84, 82)$$

$$\begin{aligned}
 \mathbf{e} \quad E(A) &= 2E(Y_1) - E(Y_2) + 3E(Y_3) \\
 &= 2 \times 8 - 12 + 15 \\
 &= 19
 \end{aligned}$$

$$\begin{aligned}
 \text{Var}(A) &= 4\text{Var}(Y_1) + \text{Var}(Y_2) + \text{Var}(Y_3) \\
 &= 4 \times 2 + 3 + 4 \\
 &= 15
 \end{aligned}$$

$$A \sim N(19, 15)$$

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Exercise A, Question 5

Question:

A , B and C are independent normal random variables.

$A \sim N(50,6)$, $B \sim N(60,8)$ and $C \sim N(80,10)$. Find:

- a $P(A+B < 115)$,
- b $P(A+B+C > 198)$,
- c $P(B+C < 138)$,
- d $P(2A+B-C < 70)$,
- e $P(A+3B-C > 140)$,
- f $P(105 < (A+B) < 116)$.

Solution:

a $A + B \sim N(50 + 60, 6 + 8) = N(110, 14)$

$$P(A + B < 115) = P\left(z < \frac{115 - 110}{\sqrt{14}}\right)$$

$$= P(z < 1.34)$$

$$= 0.9099 \text{ (0.9093)}$$

Answers which round to (awrt) 0.91

b $A + B + C \sim N(50 + 60 + 80, 6 + 8 + 10) = N(190, 24)$

$$P(A + B + C > 198) = P\left(z > \frac{198 - 190}{\sqrt{24}}\right)$$

$$= P(z < 1.63)$$

$$= 1 - 0.9484$$

$$= 0.0516 \text{ (0.0512)}$$

c $B + C \sim N(60 + 80, 8 + 10) = N(140, 18)$

$$P(B + C < 138) = P\left(z < \frac{138 - 140}{\sqrt{18}}\right)$$

$$= P(z < -0.47)$$

$$= 1 - 0.6808$$

$$= 0.3192 \text{ (0.3186)}$$

Awrt 0.319

d $2A + B - C \sim N(2 \times 50 + 60 - 80, 4 \times 6 + 8 + 10) = N(80, 42)$

$$P(2A + B - C < 70) = P\left(z < \frac{70 - 80}{\sqrt{42}}\right)$$

$$= P(z < -1.54)$$

$$= 1 - 0.9382$$

$$= 0.0618 \text{ (0.0614)}$$

e $A + 3B - C \sim N(50 + 3 \times 60 - 80, 6 + 9 \times 8 + 10) = N(150, 88)$

$$P(A + 3B - C > 140) = P\left(z > \frac{140 - 150}{\sqrt{88}}\right)$$

$$= P(z > -1.07)$$

$$= 0.8577 \text{ (0.8568)}$$

Awrt 0.858 (0.857)

f $A + B \sim N(50 + 60, 6 + 8) = N(110, 14)$

$$P(105 < A + B < 116) = P\left(\frac{105 - 110}{\sqrt{14}} < z < \frac{116 - 110}{\sqrt{14}}\right)$$

$$= P(-1.34 < z < 1.60)$$

$$= 0.9452 - (1 - 0.9099)$$

$$= 0.8551 \text{ (0.8549)}$$

Awrt 0.855

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Exercise A, Question 6

Question:

Given the random variables $X \sim N(20, 5)$ and $Y \sim N(10, 4)$ where X and Y are independent, find

- a $E(X - Y)$,
- b $\text{Var}(X - Y)$,
- c $P(13 < X - Y < 16)$.

E

Solution:

- a $E(X - Y) = 20 - 10 = 10$
- b $\text{Var}(X - Y) = 5 + 4 = 9$
- c $X - Y \sim N(10, 9)$

$$\begin{aligned} P(13 < X - Y < 16) &= P(X - Y < 16) - P(X - Y < 13) \\ &= P(z < 2) - P(z < 1) \\ &= 0.9772 - 0.8413 \\ &= 0.1359 \\ &\text{Awr } 0.136 \end{aligned}$$

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Exercise A, Question 7

Question:

The random variable R is defined as $R = X + 4Y$ where $X \sim N(8, 2^2)$, $Y \sim N(14, 3^2)$ and X and Y are independent.

Find

- $E(R)$,
- $\text{Var}(R)$,
- $P(R < 41)$.

The random variables Y_1 , Y_2 and Y_3 are independent and each has the same distribution as Y . The random variable S is defined as

$$S = \sum_{i=1}^3 Y_i - \frac{1}{2} X.$$

- Find $\text{Var}(S)$.

E

Solution:

$$\text{a } E(R) = E(X) + 4E(Y) = 8 + (4 \times 14) = 64$$

$$\text{b } \text{Var}(R) = \text{Var}(X) + 16 \text{Var}(Y) = 2^2 + (16 \times 3^2) = 148$$

$$\text{c } P(R < 41) = P\left(Z < \frac{41 - 64}{\sqrt{148}}\right) = P(Z < -1.89) = 0.0294 \text{ (0.0293)}$$

$$\begin{aligned} \text{d } S &= Y_1 + Y_2 + Y_3 - 0.5X \\ \text{Var}(S) &= 3 \text{Var}(Y) + \left(\frac{1}{2}\right)^2 \text{Var}(X) \\ &= 27 + 1 \\ &= 28 \end{aligned}$$

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Exercise A, Question 8

Question:

A factory makes steel rods and steel tubes. The diameter of a steel rod is normally distributed with mean 3.55 cm and standard deviation 0.02 cm. The internal diameter of a steel tube is normally distributed with mean 3.60 cm and standard deviation 0.02 cm.

A rod and a tube are selected at random. Find the probability that the rod cannot pass through the tube. *E*

Solution:

$$T \sim N(3.60, 0.02^2) \quad R \sim N(3.55, 0.02^2)$$

$$P(T < R) = P(T - R < 0)$$

$$E(T - R) = 3.60 - 3.55 = 0.05$$

$$\text{Var}(T - R) = 0.02^2 + 0.02^2 = 0.0008$$

$$\begin{aligned} P(T - R < 0) &= P\left(z < \frac{0 - 0.05}{\sqrt{0.0008}}\right) \\ &= P(z < -1.77) \\ &= 1 - 0.9616 \\ &= 0.0384 \quad (0.0385) \end{aligned}$$

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Exercise A, Question 9

Question:

The weight of a randomly selected tin of jam is normally distributed with a mean weight of 1 kg and a standard deviation of 12 g. The tins are packed in boxes of 6 and the weight of the box is normally distributed with mean weight 250 g and standard deviation 10 g. Find the probability that a randomly chosen box of 6 tins will weigh less than 6.2 kg. *E*

Solution:

$$T \sim N(1000, 12^2) \quad B \sim N(250, 10^2)$$

$$Y = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + B_1$$

$$E(Y) = 6 \times 1000 + 250 = 6250$$

$$\text{Var}(Y) = 6 \times 12^2 + 10^2 = 964$$

$$\begin{aligned} P(Y < 6200) &= P\left(z < \frac{6200 - 6250}{\sqrt{964}}\right) \\ &= P(z < -1.61) \\ &= 1 - 0.9463 \\ &= 0.0537 \end{aligned}$$

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Exercise A, Question 10

Question:

The thickness of paperback books can be modelled as a normal random variable with mean 2.1 cm and variance 0.39 cm^2 . The thickness of hardback books can be modelled as a normal random variable with mean 4.0 cm and variance 1.56 cm^2 . A small bookshelf is 30 cm long.

- a Find the probability that a random sample of
- 15 paperback books can be placed side-by-side on the bookshelf,
 - 5 hardback and 5 paperback books can be placed side-by-side on the bookshelf.
- b Find the shortest length of bookshelf needed so that there is at least a 99% chance that it will hold a random sample of 15 paperback books. *E*

Solution:

$$P \sim N(2.1, 0.39) \quad H \sim N(4.0, 1.56)$$

a i $Y = P_1 + P_2 + P_3 + \dots + P_{15}$

$$E(Y) = 15 \times 2.1 = 31.5$$

$$\text{Var}(Y) = 15 \times 0.39 = 5.85$$

$$\begin{aligned} P(Y < 30) &= P\left(z < \frac{30 - 31.5}{\sqrt{5.85}}\right) \\ &= P(z < -0.62) \\ &= 1 - 0.7324 \\ &= 0.2676 \\ \text{Awrt } 0.268 \end{aligned}$$

ii $Y = P_1 + P_2 + \dots + P_5 + H_1 + H_2 + \dots + H_5$

$$E(Y) = 5 \times 2.1 + 5 \times 4.0 = 30.5$$

$$\text{Var}(Y) = 5 \times 0.39 + 5 \times 1.56 = 9.75$$

$$\begin{aligned} P(Y < 30) &= P\left(z < \frac{30 - 30.5}{\sqrt{9.75}}\right) \\ &= P(z < -0.16) \\ &= 1 - 0.5636 \\ &= 0.4364 \\ \text{Awrt } 0.436 \end{aligned}$$

b $Y = P_1 + P_2 + P_3 + \dots + P_{15}$

$$E(Y) = 15 \times 2.1 = 31.5$$

$$\text{Var}(Y) = 15 \times 0.39 = 5.85$$

$$P(Y < x) > 0.99$$

$$P\left(z < \frac{x - 31.5}{\sqrt{5.85}}\right) > 0.99$$

$$\frac{x - 31.5}{\sqrt{5.85}} > 2.3263$$

$$x = 37.1$$

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Exercise A, Question 11

Question:

A sweet manufacturer produces two varieties of fruit sweet, Xtras and Yummies. The weights, X and Y in grams, of randomly selected Xtras and Yummies are such that

$$X \sim N(30, 25) \text{ and } Y \sim N(32, 16).$$

- a Find the probability that the weight of two randomly selected Yummies will differ by more than 5 g.

One sweet of each variety is selected at random.

- b Find the probability that the Yummy sweet weighs more than the Xtra.

A packet contains 6 Xtras and 4 Yummies.

- c Find the probability that the average weight of the sweets in the packet lies between 28 g and 33 g. **E**

Solution:

a $W = Y_1 - Y_2$
 $E(W) = E(Y_1) - E(Y_2) = 0$
 $\text{Var}(W) = \text{Var}(Y_1) + \text{Var}(Y_2) = 32$
 $P(|W| > 5) = P(W < -5) + P(W > 5)$
 $= P\left(z < \frac{-5 - 0}{\sqrt{32}}\right) + P\left(z > \frac{5 - 0}{\sqrt{32}}\right)$
 $= P(z < -0.88) + P(z > 0.88)$
 $= (1 - 0.8106) + (1 - 0.8106)$
 $= 0.3788 \text{ (0.3768)}$
 Awrt 0.379/0.377

b $W = Y_1 - X_1$
 $E(W) = E(Y_1) - E(X_1) = 2$
 $\text{Var}(W) = \text{Var}(Y_1) + \text{Var}(X_1) = 41$
 $P(W > 0) = P\left(z > \frac{0 - 2}{\sqrt{41}}\right)$
 $= P(z > -0.31)$
 $= 0.6217 \text{ (0.6226)}$
 Awrt 0.622/0.623

c $W = X_1 + X_2 + \dots + X_6 + Y_1 + Y_2 + \dots + Y_4$
 $E(W) = 6E(X) + 4E(Y) = 308$
 $\text{Var}(W) = 6\text{Var}(X) + 4\text{Var}(Y) = 214$

Since the total weight of the 10 sweets must be between 280 g and 330 g

$$P(28 < \bar{W} < 33) = P\left(\frac{280 - 308}{\sqrt{214}} < z < \frac{330 - 308}{\sqrt{214}}\right)$$

$$= P(-1.91 < z < 1.50)$$

$$= 0.9332 - (1 - 0.9719)$$

$$= 0.9051 \text{ (0.9059)}$$
 Awrt 0.905/0.906

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Exercise A, Question 12

Question:

If X_1, X_2, \dots, X_n , are independent random variables, each with mean μ and variance σ^2 , and the random variable Z is defined as $Z = X_1 + X_2 + \dots + X_n$, show that $E(Z) = n\mu$ and $\text{Var}(Z) = n\sigma^2$.

A certain brand of biscuit is individually wrapped. The weight of a biscuit can be taken to be normally distributed with mean 75 g and standard deviation 5 g. The weight of an individual wrapping is normally distributed with mean 10 g and standard deviation 2 g. Six of these individually wrapped biscuits are then packed together. The weight of the packing material is a normal random variable with mean 40 g and standard deviation 3 g. Find, to 3 decimal places, the probability that the total weight of the packet lies between 535 g and 565 g. **E**

Solution:

$$\begin{aligned} E(Z) &= E(X_1) + E(X_2) + \dots + E(X_n) \\ &= \mu + \mu + \dots + \mu \\ &= n\mu \end{aligned}$$

$$\begin{aligned} \text{Var}(Z) &= \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n) \\ &= \sigma^2 + \sigma^2 + \dots + \sigma^2 \\ &= n\sigma^2 \end{aligned}$$

$$B \sim N(75, 5^2) \quad W \sim N(10, 2^2) \quad X \sim N(40, 3^2)$$

$$W = B_1 + B_2 + \dots + B_6 + W_1 + W_2 + \dots + W_6 + X$$

$$E(W) = 6E(B) + 6E(W) + E(X) = 550$$

$$\text{Var}(W) = 6\text{Var}(B) + 6\text{Var}(W) + \text{Var}(X) = 183$$

$$\begin{aligned} P(535 < W < 565) &= P\left(\frac{535 - 550}{\sqrt{183}} < z < \frac{565 - 550}{\sqrt{183}}\right) \\ &= P(-1.11 < z < 1.11) \\ &= 0.8665 - (1 - 0.8665) \\ &= 0.733 \quad (0.732) \end{aligned}$$