Math 4315 - PDE's

Monday Oct 3, 2016

2nd Order Linear PDE's

$$a(x,y)\,u_{xx} + b(x,y)\,u_{xy} + c(x,y)\,u_{yy} + \text{lots} = 0$$

lots - lower order terms

we defined type

if $b^2 - 4ac > 0$ hyperbolic

$b^2 - 4ac = 0$ parabolic

$b^2 - 4ac < 0$ elliptic

we introduced 3 PDE at the cornerstone of applied Mathematics (in the variables x & y)

$u_{xx} - u_{yy} = 0$ wave eq^n - hyperbolic

$u_{xx} - u_y = 0$ heat eq^n - parabolic

$u_{xx} + u_{yy} = 0$ Laplaces eq^n - elliptic

As an example we considered

$$u_{xx} - 2u_{xy} + u_{yy} = 0$$

$b^2 - 4ac = 0$
so parabolic

and a change of variables

$$r = x + y \qquad s = y$$

and we say this PDE transformed to

$$U_{ss} = 0$$

which lead to $u = f(r)s + g(r)$

and in terms of x & y

$$u = f(x+y)y + g(x+y)$$ the general sol^n

Another example is

$$5u_{xx} - 6u_{xy} + 2u_{yy} = 0$$

Here, $a = 5$, $b = -6$, $c = 2$ so $b^2 - 4ac = 36 - 40 = -4 < 0$

so elliptic

Consider the change of variables

$$r = x + y \quad s = x + 2y$$

so $r_x = 1$, $r_y = 1$ $\quad r_{xx} = r_{xy} = r_{yy} = 0$

$s_x = 1$ $\quad s_y = 2$ $\quad s_{xx} = s_{xy} = s_{yy} = 0$

and the second order chain rules are

$$u_{xx} = u_{rr} + 2u_{rs} + u_{ss}$$

$$u_{xy} = u_{rr} + 3u_{rs} + 2u_{ss}$$

$$u_{yy} = u_{rr} + 4u_{rs} + 4u_{ss}$$

and $5u_{xx} - 6u_{xy} + 2u_{yy} = 0$ becomes

$$5u_{rr} + 10u_{rs} + 5u_{ss}$$

$$-6u_{rr} - 18u_{rs} - 12u_{ss}$$

$$+ 2u_{rr} + 8u_{rs} + 8u_{ss} = 0$$

$\Rightarrow \quad u_{rr} + u_{ss} = 0$ Laplace's Eq^{n}

So can we always transform a certain type to one of the same type but simpler.

Let us introduce standard forms:

parabolic $\quad u_{ss} + \text{lots} = 0$

hyperbolic $\quad u_{rr} - u_{ss} + \text{lots} = 0$

elliptic $\quad u_{rr} + u_{ss} + \text{lots} = 0$

can we transform to one of these standard form?

We saw in the 2 previous examples we could but how do we find the change of variables?

So in the next few lectures we will target each type

First we will assume a few things

(1) if $r = r(x, y)$, $s = s(x, y)$ is the change of variables that the Jacobian of the transformation $\neq 0$

i.e. $r_x s_y - r_y s_x \neq 0$

This way we can go back and forth between variables $(x, y) \leftrightarrow (r, s)$

Next we will assume that in the general case the type will not change. i.e.

$$x^2 u_{xx} - 3xy u_{xy} + y^2 u_{yy} = 0$$

type $b^2 - 4ac = 9x^2y^2 - 8x^2y^2 = x^2y^2 > 0$ if $xy \neq 0$

However, it is possible that $x = 0$ or $y = 0$.

Parabolic PDEs

Here we require that $b^2 - 4ac = 0$

So for the general PDE

$$a u_{xx} + b u_{xy} + c u_{yy} + \text{lots} = 0$$

we sub in our general 2nd order chain rules

$$a\left(u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx} \right)$$
$$b\left(u_{rr} r_x r_y + u_{rs}(r_x s_y + r_y s_x) + u_{ss} s_x s_y + u_r r_{xy} + u_s s_{xy} \right)$$
$$c\left(u_{rr} r_y^2 + 2u_{rs} r_y s_y + u_{ss} s_y^2 + u_r r_{yy} + u_s s_{yy} \right)$$
$$+ \text{lots} = 0$$

and regroup

$$(a r_x^2 + b r_x r_y + c r_y^2) u_{rr}$$

$$+ \left(2a r_x s_x + b(r_x s_y + r_y s_x) + 2c\, r_y s_y \right) u_{rs}$$

$$+ (a s_x^2 + b s_x s_y + c s_y^2) u_{ss} \quad + \text{lots} = 0$$

Now we choose to hit the parabolic target

$$u_{ss} + \text{lots} = 0$$

This means choosing

$$a r_x^2 + b r_x r_y + c r_y^2 = 0 \quad -(1)$$

&

$$2a r_x s_x + b(r_x s_y + r_y s_x) + 2c r_y s_y = 0 \quad -(2)$$

these at first sight are nonlinear but

$$b^2 - 4ac = 0$$

we will assume $a \neq 0$ otherwise $b = 0$

and the PDE is

$c u_{yy} + \text{lots} = 0$ and already is in standard form after dividing by c

so $c = \dfrac{b^2}{4a}$

So (1) becomes $a r_x^2 + b r_x r_y + \dfrac{b^2}{4a} r_y^2 = 0$

$$\Rightarrow \quad 4a^2 r_x^2 + 4ab r_x r_y + b^2 r_y^2 = 0$$

$$\Rightarrow \quad (2a r_x + b r_y)^2 = 0 \quad \boxed{2a r_x + b r_y = 0}$$

linear 1st order

Further from (2)

$$\overbrace{2a\, r_x s_x + b r_x s_y} + \underbrace{b r_y s_x + \frac{2 \cdot b^2}{4a} r_y s_y} = 0$$

$$\underset{\substack{\shortparallel \\ 0}}{(2a r_x + b r_y)} s_x + \frac{b}{2a} \underset{\substack{\shortparallel \\ 0}}{(2a r_x + b r_y)} s_y = 0$$

So this is automatically satisfied!

So any s will work.

Ex $\quad u_{xx} - 2u_{xy} + u_{yy} = 0 \quad$ Previous ex

$b^2 - 4ac = 0 \quad$ as before

Instead of remember the formula $2a r_x + b r_y = 0$

it's easier to go to the PDE

$$u_{xx} - 2u_{xy} + u_{yy} = 0$$

So $\quad 1 \cdot r_x^2 - 2 r_x r_y + r_y^2 = 0$

$$(r_x - r_y)^2 = 0 \quad \text{or} \quad r_x - r_y = 0$$

this factors

it always

will!

M of C $\quad \frac{dx}{1} = \frac{dy}{-1}$; $dr = 0$

$\Rightarrow \quad c_1 = x+y \quad c_2 = r$

so $\quad r = R(x+y) \quad s = S(x,y)$ anything!

We picked $r = x+y$, $s = y$

More complicated choices will create lower order terms.

Do another example before this - see pg 12

Ex $\quad x^2 u_{xx} + 4xy\, u_{xy} + 4y^2 u_{yy} = 0$

$a = x^2$, $b = 4xy$ $\quad c = 4y^2 \quad b^2 - 4ac = 16x^2y^2 - 16x^2y^2$
$= 0$

so parabolic

$x^2 r_x^2 + 4xy\, r_x r_y + 4y^2 r_y^2 = 0$

$\Rightarrow \quad x r_x + 2y r_y = 0$

M of C $\quad \frac{dx}{x} = \frac{dy}{2y}$; $dr = 0$

$\Rightarrow \quad 2\ln x - \ln y = c_1 \quad r = c_2$

so $\quad r = R(2\ln x - \ln y)$ $\quad$ OR $\quad r = R\left(\frac{x^2}{y}\right)$

So which is better

#1 let $r = 2\ln x - \ln y$ $\quad s = y$ (again anything)

$$r_x = \frac{2}{x} \quad r_y = -\frac{1}{y} \quad r_{xx} = \frac{-2}{x^2} \quad r_{xy} = 0 \quad r_{yy} = \frac{1}{y^2}$$

$$s_x = 0 \quad s_y = 1 \quad s_{xx} = s_{xy} = s_{yy} = 0$$

so $$u_{xx} = \frac{4}{x^2} u_{rr} - \frac{2}{x^2} u_r$$

$$u_{xy} = -\frac{2}{xy} u_{rr} + \frac{2}{x} u_{rs}$$

$$u_{yy} = \frac{1}{y^2} u_{rr} - \frac{2}{y} u_{rs} + u_{ss} + \frac{u_r}{y^2}$$

Now sub

$$x^2 \left(\frac{4}{x^2} u_{rr} - \frac{2u_r}{x^2} \right)$$

$$+ 4xy \left(-\frac{2}{xy} u_{rr} + \frac{2}{x} u_{rs} \right)$$

$$+ 4y^2 \left(\frac{u_{rr}}{y^2} - \frac{2}{y} u_{rs} + u_{ss} + \frac{u_r}{y^2} \right) = 0$$

$$\Rightarrow \quad (\cancel{4 - 8 + 4}^{\,0}) u_{rr} + (\cancel{8y - 8y}^{\,0}) u_{rs} + 4y^2 u_{ss}$$

$$-2u_r + 4u_r = 0$$

$$\Rightarrow \quad 4y^2 U_{ss} + 2U_r = 0$$

$$\Rightarrow \quad U_{ss} + \frac{U_r}{2y^2} = 0$$

$$\boxed{\Rightarrow \quad U_{ss} + \frac{U_r}{2s^2} = 0} \qquad \because \; y = s$$

#2 let $r = 2\ln x - \ln y$, $s = \ln y$

New eq^n is

$$U_{ss} + \frac{1}{2}U_r - U_s = 0$$

#3 $r = x^2/y \quad s = y$

$$r_x = \frac{2x}{y} \quad r_y = -\frac{x^2}{y^2} \quad r_{xx} = \frac{2}{y} \quad r_{xy} = -\frac{2x}{y^2} \quad r_{yy} = \frac{2x^2}{y^3}$$

$$s_x = 0 \quad s_y = 1 \quad s_{xx} = s_{xy} = s_{yy} = 0$$

$$U_{xx} = \frac{4x^2}{y^2} U_{rr} - \cancel{\frac{4x^3}{y^3}} + \frac{2U_r}{y}$$

$$U_{xy} = -\frac{2x^3}{y^3} U_{rr} + \frac{2x}{y} U_{rs} - \frac{2x}{y^2} U_r$$

$$U_{yy} = \frac{x^4}{y^4} U_{rr} - \frac{2x^2}{y^2} U_{rs} + U_{ss} + \frac{2x^2}{y^3} U_r$$

and sub & simplyfying gives

$$2y^3 u_{ss} + x^2 u_r = 0$$

From original tx. $r = \frac{x^2}{y}$, $y = s$

$\Rightarrow \quad x^2 = rs, \; y = s$

$\Rightarrow \quad 2s^3 u_{ss} + rs\, u_r = 0$

$\Rightarrow \quad u_{ss} + \frac{r u_r}{2s^2} = 0$

Ex 2 $\quad 4u_{xx} + 12u_{xy} + 9u_{yy} = 0$

$$4r_x^2 + 12r_x r_y + 9r_y^2 = 0$$

$$\Rightarrow \quad 2r_x + 3r_y = 0$$

M of C $\quad \frac{dx}{2} = \frac{dy}{3} \; ; \; dr = 0$

$r = R(3x-2y) \quad s$ - anything

pick $\quad r = 3x-2y \quad s = y$

so $\quad r_x = 3 \quad r_y = -2 \quad r_{xx} = r_{xy} = r_{yy} = 0$

$\quad s_x = 0 \quad s_y = 1 \quad s_{xx} = s_{xy} = s_{yy} = 0$

$u_{xx} = 9u_{rr}$

$u_{xy} = -6u_{rr} + 3u_{rs}$

$u_{yy} = \quad 4u_{rr} - 4u_{rs} + u_{ss}$

Sub $\quad 36u_{rr}$

$\quad - 72u_{rr} + 36u_{rs}$

$\quad 36u_{rr} - 36u_{rs} + 9u_{ss} = 0 \quad \Rightarrow \quad u_{ss} = 0$

$u = f(r)s + g(r)$ so $\quad u = f(3x-2y)y + g(3x-2y)$