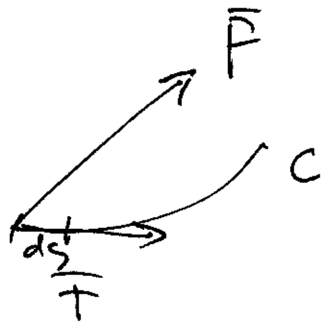
Math 2471 – Calc 3

Line integrals over VF

Consider the work done moving a particle in a VF along a curve c. Project $\bar F$ onto $\bar T$ mult. by ds then add.



$$\int_C \bar F\cdot\bar T\,ds$$

Now $\bar r = \langle x, y\rangle$ $\quad \bar r' = \langle x', y'\rangle$

so $\bar T = \dfrac{\bar r'}{\|\bar r'\|} = \dfrac{\langle x', y'\rangle}{\sqrt{x'^2+y'^2}}$

$ds = \sqrt{x'^2+y'^2}\,dt$

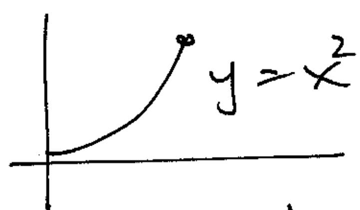
so
$$\int_C \bar F\cdot\bar T\,ds = \int \frac{\langle P, Q\rangle\cdot\langle \frac{dx}{dt}, \frac{dy}{dt}\rangle}{\cancel{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}}\cancel{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}\,dt$$

$$= \int_C \left(P\frac{dx}{dt}+Q\frac{dy}{dt}\right)\cdot dt = \int_C P\,dx+Q\,dy$$

Ex 1 if $\vec{F} = \langle y, 2x \rangle$

find $\int_C P\,dx + Q\,dy$ where C is the path along $y = x^2$ from $(0,0) \to (1,1)$

so $\int_C y\,dx + 2x\,dy$

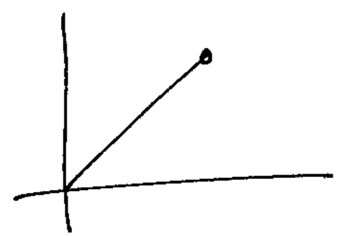


$dy = 2x\,dx$

$\int_0^1 x^2\,dx + 2x \cdot 2x\,dx$

$\int_0^1 3x^2\,dx = x^3 \Big|_0^1 = 1$

along $y = x$ $(0,0) \to (1,1)$

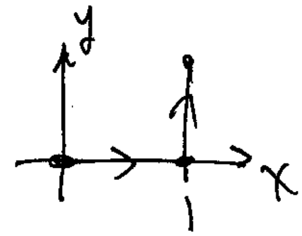


$dy = dx$

$\int_0^1 x\,dx + 2x\,dx = \int_0^1 3x\,dx = \frac{3}{2} x^2 \Big|_0^1 = 3/2$

so they are different.

Ex 2 Find $\int_C x\,dy$ when C is



Here we have 2 curves

so $\int_{C_1} x\,dy + \int_{C_2} x\,dy$

C_1 $y=0$ $x: 0 \to 1$ so $dy=0$ so $\int_{C_1} x\,dy = 0$

C_2 $x=1$ $y: 0 \to 1$ so

$$\int_0^1 1\,dy = y\Big|_0^1 = 1$$

so $\int_C x\,dy = 0+1 = 1$

Ex3 $\int_C x^3 dx - z\,dy + 2xy\,dz$

when $\bar{r}(t) = \langle t^2, \sqrt{t}, \sqrt{t} \rangle$ $2 \le t \le 4$

so $x = t^2$ $y = t^{1/2}$ $z = t^{1/2}$

$dx = 2t\,dt$ $dy = \frac{1}{2t^{1/2}} dt$ $dz = \frac{1}{2t^{1/2}} dt$

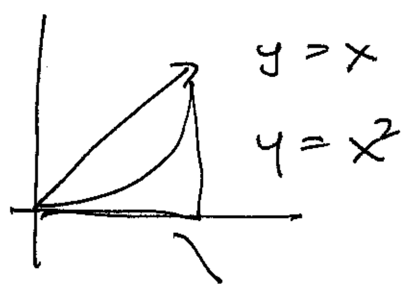
$$\int_2^4 t^6 \cdot 2t\,dt - \frac{\cancel{\sqrt{t}}\ ^1}{2\cancel{\sqrt{t}}}\,dt + \cancel{2}\cdot t^2 \cdot \frac{\cancel{\sqrt{t}}}{\cancel{2}\cancel{\sqrt{t}}}\,dt$$

$$\int_2^4 \left(2t^7 - \frac{1}{2} + t^2\right)dt = \frac{2t^8}{8} - \frac{t}{2} + \frac{t^3}{3}\Bigg|_2^4$$

$$= \left(4^7 - 2 + \frac{4^3}{3}\right) - \left(\frac{2^8}{8} - 1 + \frac{8}{3}\right) = \frac{49013}{3}$$

consider

$$\int_C 2xy\,dx + x^2\,dy$$



(i) $y = x \quad dy = dx \quad x: 0 \to 1$

$$\int 2x^2 + x^2\,dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1$$

(ii) $y = x^2 \quad dy = 2x\,dx$

$$\int_0^1 2x^3\,dx + 2x^3\,dx = \int_0^1 4x^3\,dx = x^4\Big|_0^1 = 1 \quad \text{Same}$$

C_1: $y=0$ $x: 0 \to 1$ so $\int_{C_1} 0 = 0$

C_2: $x=1$ $y: 0 \to 1$ so $\int_0^1 1\, dy = y\Big|_0^1 = 1$

so $\int_C 2xy\,dx + x^2\,dy = 1$ for all 3 curve!

So may there is something special about the vector field - its conservative!

consider

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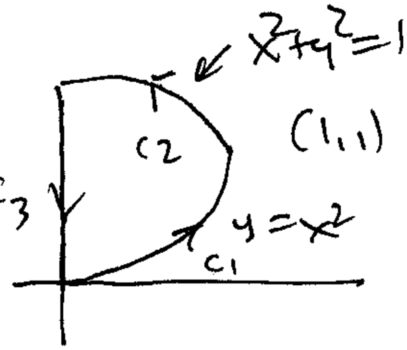
33, 35, 43, 45

$$\int_C \frac{1-x^2+y^2}{(1+x^2+y^2)^2}\,dx - \frac{2xy}{(1+x^2+y^2)^2}\,dy$$

Just $\int_C P\,dx + Q\,dy + R\,dz$

where C is



so we would need 3 separate line integral and each one (except the last one C_3)

but if it (the VF) is conservative it would certainly help (wouldn't it?)