Some Results on Slant Submanifold of Trans-Sasakian Manifolds

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Abstract:The objective of this paper is to study slant sub manifolds of trans-sasakian manifold when structure tensor field φ is killing and have obtained some results under certain conditions.

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1. Introduction

The slant submanifold for an almost Hermitian manifold is defined by Chen [8], in which generalisation of both holomorphic and totally real submanifolds are discussed. The concept of Slant submanifolds were given by Chen [2], and also the examples were discussed by Chen and Tazawa [9, 10]. In 1985, Oubina [3] introduced a new class of contact manifolds namely Trans-sasakian manifold. Many authors [4, 5, 6, 7] have studied this manifold and has obtained many interesting results. The concept of slant immersions into almost contact metric manifolds was introduced by Lotta [12, 13]. Later, the geometry of slant submanifold is studied by Cabrerizo et al. [1, 15] in more specialized settings of K-contact and sasakian manifolds. Also Gupta [11] et al. defined and studied about Slant submanifold of a Kenmotsu manifold. Killing structures on contact manifolds was studied by Blair [14].

In this paper we extend the study to find the condition for killing structure of slant submanifolds of trans-sasakian manifold.

2. Preliminaries

Let (M,g) be an almost contact metric manifold of dimension (2n + 1) equipped with structure (φ,ξ,η,g) consisting of a (1,1) tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying,

$$\varphi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0, \quad (2.1)$$

$$g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \eta(X) = g(X,\xi), g(X,\varphi Y) = -g(\varphi X,Y). \quad (2.2)$$

An almost contact metric manifold is called trans-sasakian manifold if

$$(\overline{\nabla}_X\varphi)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X,Y)\xi - \eta(Y)\varphi X], \quad (2.3)$$

$$(\overline{\nabla}_X\eta)Y = -\alpha g(\varphi X,Y) + \beta g(\varphi X,\varphi Y), \quad (2.4)$$

and

$$(\overline{\nabla}_X\xi) = -\alpha(\varphi X) + \beta(X - \eta(X)\xi), \quad (2.5)$$

where $\overline{\nabla}$ denotes the Levi-civita connection on $\overline{M}$. Let M be a n-dimensional Riemannian manifold with induced metric g isometrically immersed in $\overline{M}$. We denote by TM, the Lie algebra of vector fields on M and by $T^{\perp}M$ the set of all vector fields normal to M.

For X ∈ TM and N ∈ $T^{\perp}M$, we write

$$\varphi X = TX + NX, \quad (2.6)$$

$$\varphi V = tV + nV, \quad (2.7)$$

where TX and NX denotes the tangential and normal component of φX. Similarly tV and nV denotes the tangential and normal component of φV .

Definition: Let M be a submanifold of a trans-sasakianmanifold $\overline{M}$. For each non-zero vector X tangent to M making an angle θ(x), such that θ(x) ∈ [0,π/2], between φX and T_xM is called the slant angle of M. For vector field X ∈ Γ(TM), for x ∈ M, if the slant angle is constant, then the submanifold is also called the slant submanifold. Ifθ = 0, then the submanifold is called invariant submanifold. If θ = π/2, then it is called anti-invariant submanifold. If θ(x) ∈ (0,π/2), then it is called proper-slant submanifold.

Let ∇ be the Riemannian connection on M, then the Gauss and Weingarten formulae are given by,

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad (2.8)$$

$$\overline{\nabla}_X N = \nabla^{\perp}_X N - A_N X, \quad (2.9)$$

for any X,Y ∈ TM and N ∈ $T^{\perp}M$ of $\overline{M}$.

$\nabla^{\perp}$ is the connection in the normal bundle $T^{\perp}M$ of M, h is the second fundamental form of M and A_N is the Weingarten endomorphism associated with N. The second fundamental form h and the shape operator A related by,

$$g(h(X,Y),N) = g(A_N X,Y). \quad (2.10)$$

If T is the endomorphism defined by (2.6) then,

$$g(TX,Y) = -g(X,TY). \quad (2.11)$$

Thus T^2 which is denoted by Q is self adjoint.

$$T^2 = Q. \quad (2.12)$$

On the otherhand, Gauss and Weingarten formulae together with (2.3) and (2.6) implies that,

$$(\nabla_X T)Y = \alpha[g(X,Y)\xi - \eta(Y)X] + \beta[g(\varphi X,Y)\xi - \eta(Y)TX] + A_{NY}X + th(X,Y), (2.13)$$

$$(\nabla_X N)Y = -\beta\eta(Y)NX - h(X,TY) + nh(X,Y), \quad (2.14)$$

for any $X,Y \in TM$, A tensor field φ is said to be killing if,

$$(\bar{\nabla}_X\varphi)Y + (\bar{\nabla}_Y\varphi)X = 0. \quad (2.15)$$

3. Slant submanifold of Trans-sasakian manifold

In this section, we consider slant submanifold of Trans-sasakian manifold.

Theorem 3.1: Let M be a n dimensional slant submanifold of trans-sasakianmanifold $\bar{M}$, then$-\alpha TX + \beta[X - \eta(X)\xi] = \nabla_X\xi$, (3.1)

and, $h(X,\xi) = -\alpha NX$. (3.2)

Proof: Put $Y = \xi$ in (2.8) we have,

$$\bar{\nabla}_X\xi = \nabla_X\xi + h(X,\xi). \quad (3.3)$$

From (2.5)and (2.6) we get,

$$-\alpha TX - \alpha NX + \beta[X - \eta(X)\xi] = \nabla_X\xi + h(X,\xi). \quad (3.4)$$

Equating tangential and normal components, we get equations (3.1) and (3.2).

Hence the proof.

Theorem 3.2: Let M be a n dimensional slant submanifold of trans-sasakian manifold $\bar{M}$ such that ξ is normal to M then for any $X,Y \in TM$, we have

$$R(X,Y)\xi = \alpha(\nabla_Y T)X - \alpha(\nabla_X T)Y - 2\alpha\beta g(TY,X)\xi, \quad (3.5)$$

Where R is the curvature tensor field associated to the metric induced by $\bar{M}$ on M.

Proof: From Riemannian curvature tensor, we have

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi, \quad (3.6)$$

Taking covariant differentiation with respect to Y on eqn(3.1) , we get

$$\nabla_Y\nabla_X\xi = -\alpha(\nabla_Y T)X - \alpha T(\nabla_Y X) + \beta\nabla_Y X - \beta(\nabla_Y\eta)(X)\xi - \beta\eta(X)\nabla_Y\xi - \beta\eta(\nabla_Y X)\xi. \quad (3.7)$$

Similarly by interchanging X to Y and Y to X , we get

$$\nabla_X\nabla_Y\xi = -\alpha(\nabla_X T)Y - \alpha T(\nabla_X Y) + \beta\nabla_X Y - \beta(\nabla_X\eta)(Y)\xi - \beta\eta(Y)\nabla_X\xi - \beta\eta(\nabla_X Y)\xi. \quad (3.8)$$

Also,

$$\nabla_{[X,Y]}\xi = -\alpha T(\nabla_X Y) + \alpha T(\nabla_Y X) + \beta\nabla_X Y - \beta\nabla_Y X - \beta\eta(\nabla_X Y)\xi + \beta\eta(\nabla_Y X)\xi. \quad (3.9)$$

Substitute (3.7), (3.8) and (3.9) in (3.6) we get,

$$R(X,Y)\xi = \alpha(\nabla_Y T)X - \alpha(\nabla_X T)Y + \beta(\nabla_Y\eta)(X)\xi - \beta(\nabla_X\eta)(Y)\xi - \beta\eta(Y)\nabla_X\xi + \beta\eta(X)\nabla_Y\xi. \quad (3.10)$$

Using (2.4) and (3.1) we get,

$$R(X,Y)\xi = \alpha(\nabla_Y T)X - \alpha(\nabla_X T)Y - \alpha\beta g(\varphi Y,X)\xi + \alpha\beta g(\varphi X,Y)\xi + \alpha\beta[\eta(Y)TX - \eta(X)TY] + \beta^2[\eta(X)Y - \eta(Y)X]. \quad (3.11)$$

Using (2.6) we get,

$$R(X,Y)\xi = \alpha(\nabla_Y T)X - \alpha(\nabla_X T)Y - \alpha\beta g(TY,X)\xi + \alpha\beta g(TX,Y)\xi + \alpha\beta[\eta(Y)TX - \eta(X)TY] + \beta^2[\eta(X)Y - \eta(Y)X]. \quad (3.12)$$

If ξ is orthogonal to X,Y then we get (3.5)

Hence the proof.

Lemma 1:

Let M be a n dimensional slant submanifold of trans-sasakian manifold $\bar{M}$ with $R(X,Y)\xi = 0$, then for any $X,Y \in TM$, we have

$$(\nabla_Y T)X - (\nabla_X T)Y = 2\beta g(TY,X)\xi. \quad \textit{(3.13)}$$

Proof: Put $R(X,Y)\xi = 0$ in equation (3.5) we get (3.13).

Hence the proof.

Lemma 2 :

Let M be a n dimensional anti-invariant submanifold of trans-sasakian manifold $\bar{M}$ with $R(X,Y)\xi = 0$, then for any $X,Y \in TM$, we have

$$(\nabla_Y T)X = (\nabla_X T)Y. \quad (3.14)$$

Proof: Since M is anti-invariant submanifold of trans-sasakianmanifold $\bar{M}$,we have

$$g(\varphi X,Y) = 0.$$

Therefore from (3.11), we have

$$R(X,Y)\xi = \alpha[(\nabla_Y T)X - (\nabla_X T)Y] \quad (3.15)$$

Put $R(X,Y)\xi = 0$ in the above equation we get (3.14).

Hence the proof.

4. Slant submanifold of Trans-sasakian manifold with killing structure

In this section, we consider slant submanifold of Trans-sasakian manifold with killing structure tensor field φ of type (1,1).

From [11] we know that If M is a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0,1]$ such that

$$T^2X = -\lambda(X - \eta(X)\xi). \quad (4.1)$$

Further more, if θ is the slant angle of M, then $\lambda = \cos^2\theta$. Also for any $X,Y \in TM$,

we have,

$$g(TX,TY) = \cos^2\theta(g(X,Y) - \eta(X)\eta(Y)), \quad (4.2)$$

$$g(NX,NY) = \sin^2\theta(g(X,Y) - \eta(X)\eta(Y)). \quad (4.3)$$

Theorem 4.3.:Let M be a n dimensional slant submanifold of trans-sasakian manifold $\overline{M}$ with killing structure tensor field φ, then M is slant if,

$$\alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)TX + \eta(X)TY] = 0, \quad (4.4)$$

and

$$\eta(Y)NX + \eta(X)NY = 0. \quad (4.5)$$

Proof: Interchange X to Y and Y to X in equation (2.3)we have,

$$(\overline{\nabla}_Y \varphi)X = \alpha[g(X,Y)\xi - \eta(X)Y] + \beta[g(\varphi Y,X)\xi - \eta(X)\varphi Y]. \quad (4.6)$$

Add (2.3) and (4.6) we get,

$$(\overline{\nabla}_X\varphi)Y + (\overline{\nabla}_Y \varphi)X = \alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)\varphi X + \eta(X)\varphi Y]. \quad (4.7)$$

If φ has killing structure field then from (2.15) and (4.7), we have

$$\alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)\varphi X + \eta(X)\varphi Y] = 0. \quad (4.8)$$

Using equation (2.6), we get,

$$\alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)TX + \eta(X)TY + \eta(Y)NX + \eta(X)NY] = 0.$$

Comparing tangential and normal components in the above expression we get (4.4) and (4.5). Hence the proof.

Theorem 4.4.:Let M be a n dimensional slant submanifold of trans-sasakian manifold $\overline{M}$ with killing structure tensor field φ, then M is slant if and only if,

$$(\nabla_X Q)Y + (\nabla_Y Q)X = 0, \quad (4.9)$$

and

$$\beta[2g(X,Y)\xi + \eta(Y)X + \eta(X)Y] - 4\beta\eta(X)\eta(Y)\xi - \alpha[\eta(Y)TX + \eta(X)TY] = 0. \quad (4.10)$$

Proof: From (4.1) and (2.12), we have

$$QX = -\lambda(X - \eta(X)\xi). \quad (4.11)$$

Taking covariant differentiation for the equation (4.11) and making use of $(\nabla_X\eta)Y$ with repeated application of (4.11) we get,

$$(\nabla_X Q)Y = \lambda[-\alpha[g(TX,Y)\xi + \beta[g(X,Y)\xi - \eta(Y)\eta(X)\xi] + \eta(Y)[-\alpha(TX + NX) + \beta(X - \eta(X)\xi]]. \quad (4.12)$$

Interchange X to Y and Y to X in the equation (4.12)

$$(\nabla_Y Q)X = \lambda[-\alpha[g(TY,X)\xi + \beta[g(X,Y)\xi - \eta(X)\eta(Y)\xi] + \eta(X)[-\alpha(TY + NY) + \beta(Y - \eta(Y)\xi]]. \quad (4.13)$$

Add (4.12) and (4.13) we get ,

$$(\nabla_X Q)Y + (\nabla_Y Q)X = \lambda[2\beta[g(X,Y)\xi - \eta(X)\eta(Y)\xi] + \eta(Y)[-\alpha(TX + NX) + \beta(X - \eta(X)\xi] + \eta(X)[-\alpha(TY + NY) + \beta(Y - \eta(Y)\xi]]. \quad (4.14)$$

If

$$(\nabla_X Q)Y + (\nabla_Y Q)X = 0.$$

Then by using equation (4.5) we obtain,

$$\beta[2g(X,Y)\xi + \eta(Y)X + \eta(X)Y] - 4\beta\eta(X)\eta(Y)\xi - \alpha[\eta(Y)TX + \eta(X)TY] = 0.$$

Conversely, Suppose

$$\beta[2g(X,Y)\xi + \eta(Y)X + \eta(X)Y] - 4\beta\eta(X)\eta(Y)\xi - \alpha[\eta(Y)TX + \eta(X)TY] = 0.$$

then using (4.10) and (4.5) in (4.14), we get (4.9).

Hence the proof.

Theorem 4.5.: Let M be n dimensional slant submanifold of trans-sasakian manifold $\overline{M}$ with φ as a killing structure, then

$$-h(X,TY) - h(Y,TX) + 2nh(X,Y) = 0, \quad (4.15)$$

If and only if,

$$(\nabla_X N)Y + (\nabla_Y N)X = 0. \quad (4.16)$$

Proof: Interchange X to Y and Y to X in (2.14), we get

$$(\nabla_Y N)X = -\beta\eta(X)NY - h(Y,TX) + nh(X,Y). \quad (4.17)$$

Add equation (2.14) and (4.17) we have,

$$(\nabla_X N)Y + (\nabla_Y N)X = -\beta\eta(Y)NX - h(X,TY) - \beta\eta(X)NY - h(Y,TX) + 2nh(X,Y). \quad (4.18)$$

If

$$(\nabla_X N)Y + (\nabla_Y N)X = 0,$$

Then from (4.18) we get,

$$-\beta[\eta(Y)NX + \eta(X)NY] - h(X,TY) - h(Y,TX) + 2nh(X,Y) = 0. \quad (4.19)$$

Using equation (4.5) we obtain,

$$-h(X,TY) - h(Y,TX) + 2nh(X,Y) = 0.$$

Conversely, Suppose

$$-h(X,TY) - h(Y,TX) + 2nh(X,Y) = 0.$$

Then using (4.15) and (4.5) in (4.18), we get (4.16).

Hence the proof.

Theorem 4.6.: Let M be n dimensional slant submanifold of trans-sasakian manifold $\overline{M}$ with killing structure tensor field φ, then M is slant if T satisfies

$$(\nabla_X T)Y + (\nabla_Y T)X = 0, \quad (4.20)$$

then

$$A_{NY} X + A_{NX}Y + 2th(X,Y) = 0. \quad (4.21)$$

Proof: Replace X by Y and Y by X in (2.13) we get,

$$(\nabla_Y T)X = \alpha[g(X,Y)\xi - \eta(X)Y] + \beta[g(\varphi Y,X)\xi - \eta(X)TY] + A_{NX}Y + th(X,Y), \quad (4.22)$$

Add equations (2.13) and (4.22) we get,

$$(\nabla_X T)Y + (\nabla_Y T)X = \alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] + \beta[g(\varphi X,Y)\xi - \eta(Y)TX + g(\varphi Y,X)\xi - \eta(X)TY] + A_{NY} X + A_{NX}Y + 2th(X,Y). \quad (4.23)$$

If

$$(\nabla_X T)Y + (\nabla_Y T)X = 0.$$

Then we have,

$$\alpha[2g(X,Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)TX + \eta(X)TY] + A_{NY} X + A_{NX}Y + 2th(X,Y) = 0 \quad (4.24)$$

Using the equation (4.4) we get,

$$A_{NY} X + A_{NX}Y + 2th(X,Y) = 0.$$

Conversely, Suppose

$$A_{NY} X + A_{NX}Y + 2th(X,Y) = 0.$$

Then using (4.21) and (4.4) in (4.23), we get (4.20).

Hence the proof.

5. Conclusion

In this paper we have obtained some results on slant submanifolds of trans-sasakian manifold obeying certain conditions with φ as a killing structure tensor field.

References:

[1] J. L. Cabrerizo, A. Carriazo and L. M. Fernandez, Slant Submanifolds in sasakian manifolds, Glasgow Math. J.,42 (2000), 125-138.
[2] B.Y. Chen, Geometry of submanifolds, Marcel Dekker, 1973.
[3] J.A.Oubina, New classes of almost contact metric structure, Publications Mathematicae Debrecen, 32(3-4)(1985), PP.187-193.
[4] C.S.Bagewadi and E GirishKumar : Note on Trans-sasakian manifolds, Tensor N.S. 65(1)(2004), 80-88.
[5] D.E.Blair, Contact Manifolds in Riemannian geometry, Lecture notes in Mathematics, Springer Verlag, Berlin-Newyork, 509(1970).
[6] H.G.Nagaraja, R.C.Premalatha and G.Somashekhara,On an (ε,δ) - Trans sasakian structure, Proceedings of the Estonian Academy of Sciences, 61(1)(2012), PP.20-23.
[7] ViqarAzam Khan and Meraj Ali Khan,Semi-slant submanifolds of Trans-sasakian manifolds, Sarajevo Journal of Mathematics, 2(14), 83-93.
[8] B. Y. Chen, Slant immersions, Bull. Aust. Math. Soc.,41 (1990), 135-147.
[9] B. Y. Chen and Y. Tazawa, Slant surfaces with codimension 2, Ann. FAc. Sci. Toulouse Math., XI(3) (1990), 29-43.
[10] B. Y. Chen and Y. Tazawa, Slant submanifolds in complex Euclidean spaces, Tokyo J. Math., 14(1) (1991), 101-120.
[11] R. S. Gupta, S. M. KhursheedHaider and M. H. Shahid, Slant submanifolds of a Kenmotsu manifold, Radovi Matematicki,12 (2004), 205-214.
[12] A. Lotta, Slant submanifolds in contact geometry, Bull. Math. Soc. Sci. Roum., 39 (1996), 183-198.
[13] A. Lotta, Three dimensional slant submanifolds of K-contact manifolds, Balkan J. Geom. Appl., 3(1) (1998), 37-51.
[14] D. E. Blair, Almost contact manifolds with killing structure tensors, Pacific J. Math., 39(2) (1971), 285-292.
[15] J. L. Cabrerizo, A. Carriazo and L. M. Fernandez, Structure on a slant submanifold of a contactmanifold, Indian J. Pure and Appl. Math.,31(7) (2000), 857-864.