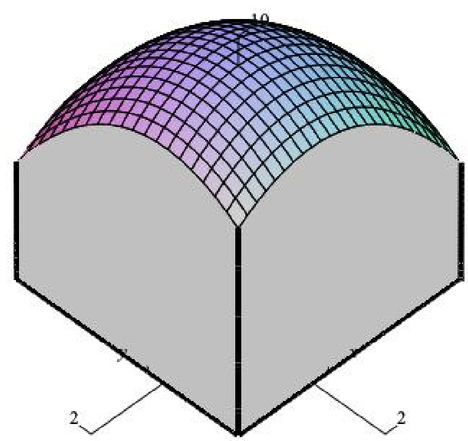
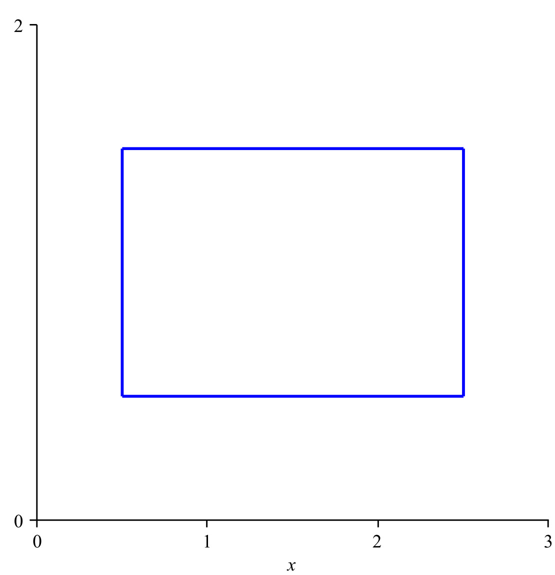
Calculus 3 - Iterated Integrals

Last class we consider the problem of finding the volume under $z = f(x,y)$ on the rectangular region $[a,b] \times [c,d]$



were we introduced the double integral

$$\int_a^b \int_c^d f(x,y)\, dy dx \tag{1}$$



noting that to evaluate the integral, we first hold one variable fixed and integrate with respect to the other and in this case hold x fixed and integrate with respect to y

$$\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx \tag{2}$$

and then integrate with respect to x.

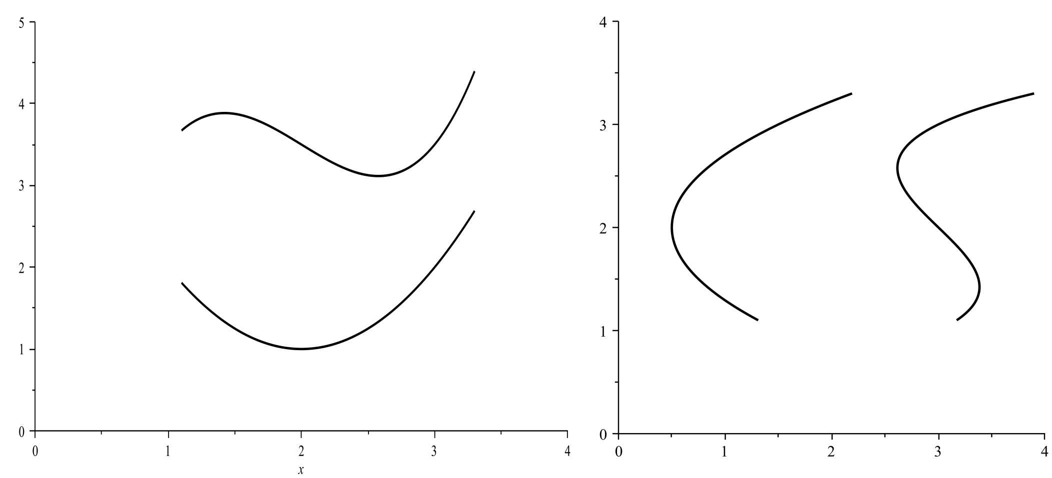
We could also reverse the order of integration and integrate first with respect to x (holding y constant)

$$\int_c^d \int_a^b f(x,y)\, dxdy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy \tag{3}$$

and then integrate with respect to y.

In general we have

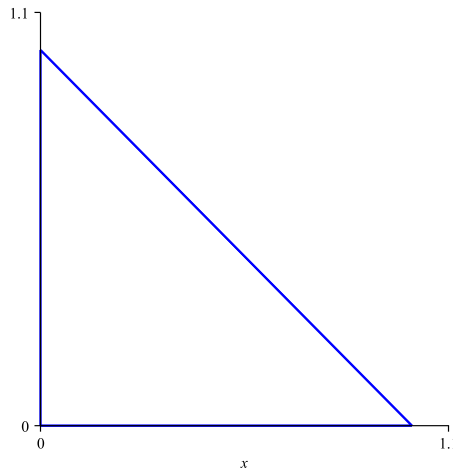
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y) dydx \qquad\qquad \int_{y=c}^{y=d} \int_{x=G(y)}^{x=H(y)} f(x,y) dxdy \tag{4}$$

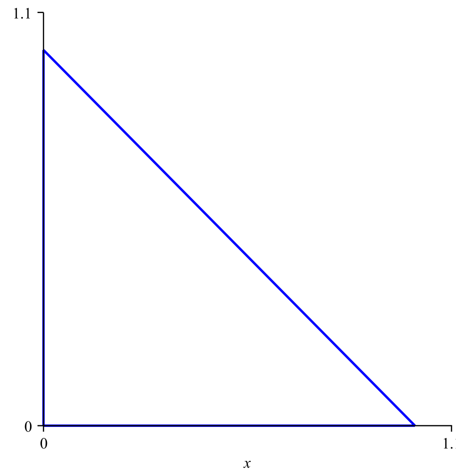


As an example, let us set up and evaluate the integral

$$\iint\limits_R 12x^2y\,dA \tag{5}$$

where $dA = dydx$ and where R is the region bound by $x + y = 1$, $x = 0$ and $y = 0$. We first sketch region





$$\int_?^? \int_?^? 12x^2ydydx \qquad\qquad \int_?^? \int_?^? 12x^2ydxdy \tag{6}$$

and here

$$\int_0^1 \int_0^{1-x} 12x^2ydydx \qquad\qquad \int_0^1 \int_0^{1-x} 12x^2ydxdy \tag{7}$$

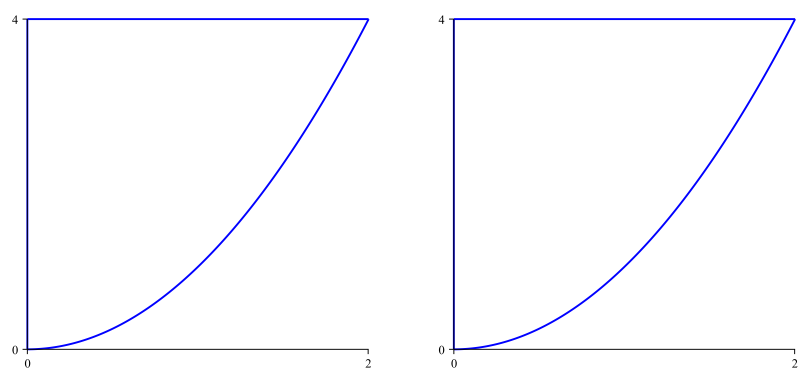
So, which one is easier to integrate?

$$
\begin{aligned}
V &= \int_0^1 \int_0^{1-x} 12x^2 y dy dx \\
&= \int_0^1 6x^2y^2 \Big|_0^{1-x} dx \\
&= \int_0^1 6x^2(1-x)^2 \, dx \\
&= \int_0^1 6x^2 - 12x^3 + 6x^4 \, dx \\
&= 2x^3 - 3x^4 + \frac{6}{5}x^5 \Big|_0^1 \\
&= 2 - 3 + \frac{6}{5} = \frac{1}{5}
\end{aligned} \tag{8}
$$

Example 2. Set up and evaluate

$$
\iint\limits_R \frac{y}{x+1} dA \tag{9}
$$

where R is the region bound by $y = x^2$, $y = 4$ and $x = 0$. We first sketch the region so



$$
\int_?^? \int_?^? \frac{y}{x+1} dy dx \qquad\qquad \int_?^? \int_?^? \frac{y}{x+1} dx dy \tag{10}
$$

In this case

$$\int_0^2 \int_{x^2}^4 \frac{y}{x+1} dydx \qquad\qquad \int_0^4 \int_0^{\sqrt{y}} \frac{y}{x+1} dxdy \tag{11}$$

Which is easier to integrate?

$$\frac{1}{2}\int_0^2 \frac{y^2}{x+1}\bigg|_0^{x^2} dx \qquad\qquad \int_0^4 y\ln(x+1)\bigg|_0^{\sqrt{y}} dy \tag{12}$$

then

$$= \frac{1}{2}\int_0^2 \frac{x^4}{x+1} dx \qquad\qquad = \int_0^4 y\ln(\sqrt{y}+1)dy \tag{13}$$

Reversing the Order of Integration

Sometimes we are given the integral already with limits and are asked to change or reverse the order of integration. For example, suppose we are given (pg 977, # 54)

$$\int_0^9 \int_{\sqrt{x}}^3 dydx \tag{14}$$

and are asked to create a double integral

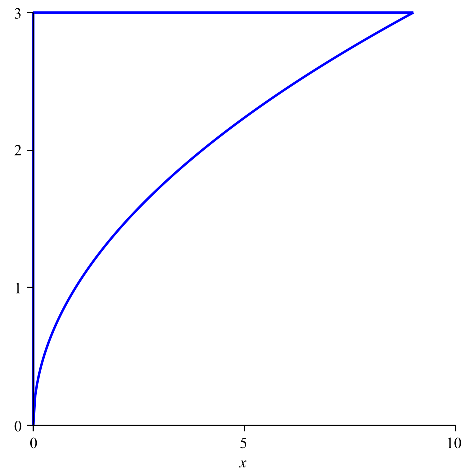
$$\int_?^? \int_?^? dxdy \tag{15}$$

Since the inside integral is with respect to y so

$$y = \sqrt{x} \quad \rightarrow \quad y = 4. \tag{16}$$

These two curves we can draw. The outside integral is point to point in the x direction and so

$$x = 0 \to x = 9. \tag{17}$$



$$\int_0^3 \int_0^{y^2} dxdy \tag{18}$$

Sometimes it is necessary to reverse the order of integration to actual integrate. The following demonstrates (pg 977, # 64). Integrate the following:

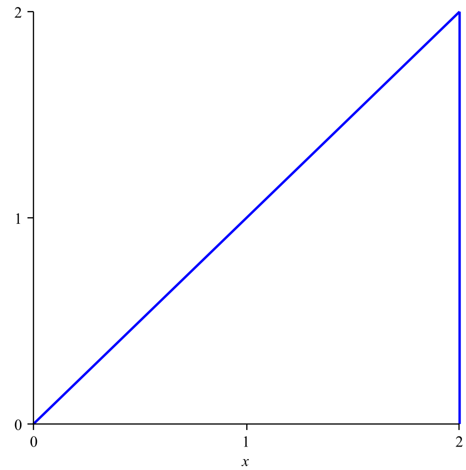
$$\int_0^2 \int_y^2 e^{-x^2} dxdy. \tag{19}$$

As it stands right now, we integrate so we will reverse of the order of integration. Since the inside integral is with respect to x (left curve to right curve) so

$$x = y \quad \to \quad x = 2. \tag{20}$$

These two curves we can draw. The outside integral is point to point in the y direction and so

$$y = 0 \quad \to \quad y = 2. \tag{21}$$



$$\begin{aligned}\int_0^2 \int_0^x e^{-x^2} dy dx &= \int_0^2 e^{-x^2} y \Big|_0^x dx \\ &= \int_0^2 x e^{-x^2} dx \quad (u = -x^2) \\ &= -\frac{1}{2} e^{-x^2} \Big|_0^2 \\ &= \frac{1}{2} \left(1 - e^{-4}\right)\end{aligned} \tag{22}$$