# Math 3331 – ODE's

so far we ~~have~~ considered $1^{st}$ order ODE's

$$y' = f(x, y)$$

and introduced ways to solve these

(1) sep. (2) linear (3) Bernoulli

(4) Ricatti (5) homogeneous (6) Exact

Now we turn our attention to solving $2^{nd}$ order ODE's

$$y'' = f(x, y, y')$$

$1^{st}$ up – Reducable $2^{nd}$ order ODEs

consider $y'' + y' = 0$

or $y'' + y = 0$

these are of the form

$$a(x)y'' + b(x)y' + c(x)y = g(x)$$

$2^{nd}$ order linear

also $y'' + y'^2 = x$, $y'' + \sin(y') = e^x$ no $y$ 

and $y'' + y'^2 = y$ $y'' + \sin(y') = e^y$ no $x$.

Can we actually solve ODE's of the form

$y'' = f(x, y')$ no $y$ ?

$y'' = f(y, y')$ no $x$ ?

Well we can actually reduce the order and turn the ODE's into 1st order. The following example illustrate

ex $y'' + \frac{1}{x}y' = 6$ (no $y$)

let $y' = u$ so $y'' = u'$

so $u' + \frac{1}{x}u = 6$ linear

$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$ so $x\frac{du}{dx} + u = 6x$

so $\frac{d}{dx}(xu) = 6x$

$d(xu) = 6x\,dx \Rightarrow xu = \frac{6x^2}{2} + c_1$

$\Rightarrow u = 3x + \frac{c_1}{x}$

Now $u = y'$ so $y' = 3x + \frac{c_1}{x}$ (sep)

$dy = 3x + \frac{c_1}{x}\,dx$

$$\boxed{y = \frac{3x^2}{2} + c_1 \ln|x| + c_2}$$

note: 2 const of integration

Ex2 $y'' = xy'^2$ (no y)

let $y' = u$ so $y'' = u'$

so $\frac{du}{dx} = xu^2$ sep $\frac{du}{u^2} = x\,dx$

$-\frac{1}{u} = \frac{x^2}{2} + \frac{c}{2}$ ( c is arb. so $\frac{c}{2}$ is as well so $u = \dfrac{-2}{x^2+C}$

Now this integral will depend on C

$\dfrac{dy}{dx} = \dfrac{-2}{x^2+C}$

Ex $\displaystyle\int \frac{dx}{x^2+1}\,dx = \tan^{-1}x + C_2$

so $y = -2\displaystyle\int \frac{dx}{x^2+C}$

$\displaystyle\int \frac{dx}{x^2}\,dx = -\frac{1}{x} + C_2$

$\displaystyle\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

so we would need more information. For example suppose we were given

$y(0) = 1$ and $y'(0) = -2$

then if $y' = \dfrac{-2}{x^2+C}$ $\quad -2 = \dfrac{-2}{0^2+C} \Rightarrow C = 1$

$\dfrac{dy}{dx} = \dfrac{-2}{x^2+1}$ $\quad y = -2\tan^{-1}(x) + C_2$

$y(0) = 1 \Rightarrow 1 = -2\tan^{-1}(0) + C_2 \Rightarrow C_2 = 1$

$\boxed{y = -2\tan^{-1}(x) + 1 \quad \text{sol}^{n}}$

Now we consider an example with no $x$

Ex 3 $\quad \frac{d^2y}{dx^2} = y$

So again we try $y' = u$ so $y'' = \frac{du}{dx}$

so $\frac{du}{dx} = y$ problem? $x, y$ & $u$

so a trick - $\quad \frac{du}{dx} = \frac{du}{dy}\underbrace{\frac{dy}{dx}}_{u} = u\frac{du}{dy}$

so $\quad u\frac{du}{dy} = y$ (now 1st order sep!

$u\,du = y\,dy$

$\Rightarrow \frac{u^2}{\cancel{2}} = \frac{y^2}{\cancel{2}} + \frac{c_1}{\cancel{2}}$ (again const of integration)

$u = \sqrt{y^2+c_1}$ so $\frac{dy}{dx} = \sqrt{y^2+c_1}$

$\frac{dy}{\sqrt{y^2+c_1}} = dx \Rightarrow \ln|y+\sqrt{y^2+c_1}| = x + c_2$

$\Rightarrow$ solve for $y$: $y = \frac{c_2}{2}e^{x} - \frac{c_1}{2}e^{-x}$

<u>Ex 4</u> $y'' = 2y' - \frac{y'^2}{y}$ (no $x$)

if $y' = u$ then $y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \cancel{\phi}\, u\frac{du}{dy}$

so $u\frac{du}{dy} = 2u - \frac{u^2}{y}$

$u = 0$ ( $y' = 0$ $y = c_1$ not very interesting )

$\frac{du}{dy} + \frac{u}{y} = 2$ linear

$\mu = e^{\int \frac{1}{y} dy} = y$

$\frac{d}{dy}(yu) = 2y \Rightarrow yu = y^2 + c_1$

$y\frac{dy}{dx} = y^2 + c_1 \Rightarrow \frac{y\,dy}{y^2 + c_1} = dx$

$\ln|y^2 + c_1| = 2x + c_2$

$y^2 = K_1 e^{2x} + K_2$ new constants!