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Research Article

Orthogonality of Finite Rank Generalized Derivations

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Abstract

Let H be an infinite dimensional Hilbert space. In this paper, we employ operator techniques, polar decomposition, Halmos generalization formula and derivation inequalities to establish orthogonality in normed spaces. An operator A is hyponormal and B^* is m-hyponormal if T is a generalized nilpotent hyponormal operator. Properties of operators in the closure of the range of the inner derivation have been used to establish orthogonality of finite rank derivations.

Keywords: Hyponormal operators; Generalized derivations; Commutator; Orthogonality; Finite rank.

Introduction

In the present work, authors presented range-kernel orthogonality conditions for finite rank generalized derivations implemented by hyponormal operators. As an active area of research, studies on derivations and range-kernel orthogonality has attracted many mathematicians [1-4]. Different notions of orthogonality; Pythagorean, isosceles, Roberts, Birkhoff, James and a Carlsson have been dealt by a number of mathematicians [5,6]. Here orthogonality is defined in [7] in the sense of Birkhoff where x ϵ H is said to be Birkhoff orthogonal to y ϵ H if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$. Operator norm for generalized derivation, $\|\delta_{A,B}(X)\| = \inf\{\|A - \lambda\| + \|B - \lambda\|\}$ where $\lambda \in \mathbb{C}$ [8] and for normal operators. In [9] they characterized generalized derivation with orthogonality by establishing the inequality; $\|(AX - XB) + T\| \geq \|T\|$ for all X ϵ B(H) [10] which implies range-kernel orthogonality for generalized derivation. The same inequality has been established by [11] in three different conditions;

i) B is invertible

ii) A is isometric and B is a contraction

iii) A is a contraction and B is co-isometric.

With respect to unitarily invariant norms [12] established $\|(AX - XB) + T\|_p \geq \|T\|_p$ for a pair (A, B) having $(FP)_{c_p}$ property i.e. Fuledge-Putnam property restricted to Schatten p-class and if the pair (A, B) has the $(FP)_{c_2}$ property, then $\|(AX - XB) + T\|_2 \geq \|T\|_2$ for all X ϵ B(H) and for all $T \in \ker\delta_{A,B} \cap C_2$. Using polar decomposition of T i.e $T = U|T|$ where U is a partial isometry such that $\ker U = \ker|T|$, [13] established $\|(AX - XB) + T\|_\infty \geq \|T\|_\infty$ which guarantees range-kernel orthogonality restricted to a pair of compact operators with $(FP)_{K(H)}$. By algebraic direct sums of $A = A_1 \oplus A_2$ with respect to $H = H_1 = \overline{R(T)} \oplus \overline{R(T)}^\perp$ and $B = B_1 \oplus B_2$ with respect to $H = H_2 = \ker(T)^\perp \oplus \ker(T)$, [14] established a similar inequality. For non-normal operators with (FP) property, [15] used orthogonal decomposition of H to establish range-kernel orthogonality inequality for self-commutator operators restricted to $\mathbb{C}_p$ classes, where $\mathbb{C}_p$ is the von Neumann Schatten p-class through global minimization.

The properties of a class of operators, $\mathfrak{T} \in \overline{R(\delta_A)} \cap \{A\}^\#$ where $\{A\}^\#$ is the commutator of $A \in F_H^{(H)}$, are instrumental in operator theory and has been of interest to investigate the properties of operators in $\overline{R(\delta_A)}$ and further establish quasinilpotent operators in $\overline{R(\delta_A)} \cap \{A\}^\#$. With this interest Mecheri [16] proved that every operator in $\overline{R(\delta_A)} \cap \{A\}^\#$ is nilpotent if P(A) is normal, isometric or co-isometric for some polynomial P and in particular every normal operator in $\overline{R(\delta_A)} \cap \{A\}^\#$ vanish and also if A ϵ B(H) satisfies one of the following conditions:

(1). A is sub-normal and has a cyclic vector

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(2). A is isometric i.e. $A^*A = I$. Then $\overline{R(\delta_A)} \cap \{A^*\}^\# = \{0\}$. Furthermore, for normal operators [17] established a sufficient condition under which $\overline{R(\delta_A)} \cap \ker\delta_{A,B} = \{0\}$ and a sufficient condition for which $\overline{R(\delta_A)} \cap \ker\delta_{A^*,B^*} = \{0\}$.

Hyponormal operators being a larger class of operators contains finite, normal and log-hyponormal operators. Range-kernel orthogonality conditions for such class of operators have been established via operator matrices, polar decomposition and minimization procedures. For instance if $A \in B(H)$ is k-quasihyponormal operator and $B^* \in B(H)$ is an injective p-hyponormal operator, then by Bachir [18] $R(\delta_{A,B})$ is orthogonal to $\ker(\delta_{A,B})$.

To establish the required result we use fundamental inequality tools due to Anderson for normal operators [19], orthogonal decomposition [20], algebraic direct sum of operators [21], algebraic properties of projections and their adjoint operators [22], matrix decomposition of operators [23], computational skills and techniques to establish range-kernel orthogonality inequalities for finite rank hyponormal operators. We take $F_H^{(H)}$ to denote the algebra of all finite rank hyponormal operators acting on an infinite dimensional Hilbert space H, $\{A\}^\#$ the commutator of $A \in F_H^{(H)}$, $R(\delta_{A,B})$ to denote the range of A, $B \in F_H^{(H)}$ and $\ker(\delta_{A,B})$ their respective kernel.

Methodology

Preliminaries

In this section, we start by defining some key terms that are useful in this paper.

Definition 2.1 ([18], Definition 1.2.26). The rank of operator A is the dimension of its range. A finite rank operator is a bounded linear operator between Banach spaces whose range is of finite dimention.

Definition 2.2 ([14], Definition 2.1). Orthogonalities:

Let $x, y \in H$ be vectors then;

(i). x is in general orthogonal to y written as $x \perp y$, if $\langle x,y\rangle = 0$

(ii). x is Birkhoff orthogonal to y denoted as $x \perp_B y$ if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

(iv). x is Roberts orthogonal to y denoted as $x \perp_R y$ if $\|x + \lambda y\| = \|x - \lambda y\|$ for all $\lambda \in \mathbb{C}$.

(v). x is isosceles orthogonal to y denoted as $x \perp_i y$ if $\|x + y\| = \|x - y\|$

(vi). x is James orthogonal to y denoted as $x \perp_j y$ if $\|y + \lambda x\| \geq \|x\|$ for all $\lambda \in \mathbb{C}$.

(vii). x is Singer orthogonal to y denoted as $x \perp_S y$ if $x = 0$ and $y = 0$.

Definition 2.3 ([1], Definition 1.3.3) The orthogonal complement $A^\perp$ of a subset A is the set of vectors orthogonal to A i.e $A^\perp = x \in H : x \perp y \ \forall\ y \in A$. Subsets A and B of H are orthogonal written as $A \perp B$ if $x \perp y$ for every $x \in A$ and $y \in B$.

Definition 2.4 ([11], Theorem 4) Let B(H) be the algebra of all bounded linear operators acting on Hilbert space, H. The mapping $\delta_{A,B} : B(H) \longrightarrow B(H)$ is called generalized derivation defined as $\delta_A(X) = AX - XB$.

Definition 2.5 ([12], Definition 4.5) Let H be a Hilbert space and B(H) be equipped with the operator norm. The operator $\delta_{A,B}$ defined on the Banach space B(H) is equipped with the operator norm $\|\delta_{A,B}X\| = \inf\{\|A - \lambda\| + \|B - \lambda\|\}$ for all λ $\mathbb{C}$ and for all $X \in B(H)$.

Definition 2.5 ([13], Definition 3.10) A bounded operator T on a Hilbert space H is said to be trace class (or lies in C_1) if $tr|T| < \infty$ where trace of T is defined as $trT = \sum_{e\in\mathcal{E}}\langle Te, e\rangle$ for some orthonormal basis $\mathcal{E}$. Furthermore if $trT < \infty$ then T is compact.

Definition 2.6 ([13], Definition 78) Let $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ be singular values of a compact operator $T \in B(H)$ arranged in their decreasing order. Then T is said to be belong to the Schatten p-class, C_p i.e $\|T\|_p = (\sum_{i=1}^{\infty} s_i(T)^p)^{\frac{1}{p}} = (tr|T|^p)^{\frac{1}{p}} < \infty$ for $1 \leq p < \infty$.

Definition 2.7 ([14], Definition 1.8.5) Let H_1 and H_2 be both Hilbert spaces. $A \in B(H_1)$ and $B \in B(H_2)$ are called unitarily equivalent, if there exist a linear unitary map U of H_1 into H_2 such that $A = U^*BU$.

Definition 2.7 ([16], Definition 3.1) Let A, $B \in B(H)$. The pair (A, B) is said to possess the

Fuglede-Putnam property , $(FP)_{B(H)}$, if AX = XB implies $A^*X = XB^*$ for all X ϵ B(H).

Definition 2.7 ([19], Definition 12.11) If T is an operator on Hilbert space H and T^* is the respective adjoint then:

i) T is normal if $TT^* = T^*T$

ii) T is self adjoint or Hermitian if $T = T^*$

iii) T is unitary if $TT^* = I = T^*T$

iv) T is idempotent if $T^2 = T$

v) T is nilpotent if $T^n = 0$ for all n ϵ ℕ

vi) T is a projection if $T^2 = T$ and $T^* = T$

vii) T is binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$

viii) T is hyponormal if $TT^* \leq T^*T$

ix) T is semi-normal if $TT^* \leq T^*T$ or $TT^* \geq T^*T$ i.e. either T or T^* is hyponormal.

x) T is quasinormal if it commutes with T^*T i.e. $T(T^*T) = (T^*T)T$.

Definition 2.7 ([20], Definition 4) An operator T ϵ B(H) is called dominant if $R(T - \lambda) \subset R(T - \lambda)^*$ for all λ ϵ ℂ i.e. if there is a real number $m_\lambda \geq 1$ such that $\|(T - \lambda)^* x\| \leq m_\lambda \|(T - \lambda)x\|$ for all x ϵ H, consequently if there is a constant k such that $m_\lambda \leq k$ for all λ, T is called m-hyponormal and if m=1, then T is hyponormal.

Results and discussion

In this section we give the main results. First we establish the necessary and sufficient conditions for orthogonality of the range and kernel of finite rank generalized derivations and then we establish range-kernel orthogonality of finite rank derivations implemented by hyponormal operators.

At this juncture, we establish orthogonality of the range and kernel of finite rank generalized derivations.

Theorem 3.1 Let A, B ϵ F_H (H) be hyponormal operators. Suppose there exist T ϵ F_H (H) such that AT = TB, then for all X ϵ $F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all T ϵ $\ker\delta_{A,B}$.

Proof. We apply Halmos [12] generalization formula for derivation;

$$A^nX - XB^n + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XB))A^i = nA^{n-1}T \quad (1)$$

For n = 1, we have AX – XB + T – (AX – XB) = T

$\Longrightarrow \| AX - XB\| + \|T - (AX - XB)\| \geq \|T\|$

For n = 2 we have $A^2X - XB^2 + 2(T - (AX - XB))A = 2AT$

$\Longrightarrow \| A^2X - XB^2\| + 2\|A\| \|T - (AX - XB)\| \geq 2\|A\|\|T\|$

$\Longrightarrow \frac{\|A^2X - XB^2\|}{2\|A\|} + \|T - (AX - XB)\| \geq \|T\|$

Similarly for n = 3 we have; $\| A^3X - XB^3\| + 3\|A^2\| \|T - (AX - XB)\| \geq 3\|A^2\|\|T\|$

$\Longrightarrow \frac{\|A^3X - XB^3\|}{3\|A^2\|} + \|T - (AX - XB)\| \geq \|T\|$

Hence for an arbitrary n ϵ ℕ we have; $\frac{\|A^nX - XB^n\|}{n\|A^{n-1}\|} + \|T - (AX - XB)\| \geq \|T\|$

Taking n → ∞ we have; $\|T - (AX - XB)\| \geq \|T\|$

Theorem 3.2 Let A, B ϵ F_H (H) be hyponormal operators such that A is contractive. Suppose there exist T ϵ F_H (H) such that AT = TB. Then for all X ϵ $F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all T ϵ $\ker\delta_{A,B}$.

Proof. Equality (1) can be written as;

$$A^nX - XB^n + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XB))A^i = nA^{n-1}T$$

For n = 1, we have AX – XB + T – (AX – XB) = T

$\Longrightarrow \| AX - XB\| + \|T - (AX - XB)\| \geq \|T\|$

For n = 2 we have $A^2X - XB^2 + 2(T - (AX - XB))A = 2AT$

$\Longrightarrow \| A^2X - XB^2\| + 2\|T - (AX - XB)\| \geq 2\|T\|$

$\Longrightarrow \frac{\|A^2X - XB^2\|}{2} + \|T - (AX - XB)\| \geq \|T\|$

Similarly for n = 3 we have; $\| A^3X - XB^3\| + 3\|T - (AX - XB)\| \geq 3\|T\|$

$\frac{\|A^3X - XB^3\|}{3} + \|T - (AX - XB)\| \geq \|T\|$

For an arbitrary n ϵ ℕ we have; $\frac{\|A^nX - XB^n\|}{n} + \|T - (AX - XB)\| \geq \|T\|$

Taking n → ∞ we have; $\|T - (AX - XB)\| \geq \|T\|$

Theorem 3.3 Let A, B ϵ F_H (H) be hyponormal operators. Suppose there exist T ϵ F_H (H) such that AT = TB, TA = BT and T is unitarily equivalent to an isometric operator S ϵ B(H). Then for all X ϵ $F_H^{(H)}$ we have $\|\mathcal{T} - (\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{B})\| \geq \|\mathcal{T}\|$ for all $\mathcal{T}$ ϵ $\ker\delta_{\mathcal{A},\mathcal{B}}$.

Proof. First we show that T is also isometric. Since S and T are unitarily equivalent it implies existence of a unitary operator U such that $T = U^*SU$

$\Longrightarrow T^*T = (U^*S^*U)(U^*SU) = U^*S^*SU$

But S is isometric, implying $S^*S = I$ and hence $T^*T = U^*U = I$, implying T is also isometric.

Using [17, Theorem 3.3] we let $\mathcal{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $\mathcal{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ be the polar decomposition of $\mathcal{T}$, $\mathcal{A}$ and $\mathcal{X}$ on H ⊕ H. T is an isometry on H implying $\mathcal{T} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ is also an isometry on $H \oplus H$ and also

$\mathcal{T}\mathcal{A} = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & TB \\ TA & 0 \end{pmatrix} = \begin{pmatrix} 0 & AT \\ BT & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} = \mathcal{A}\mathcal{T}$

$\Longrightarrow \mathcal{T}\mathcal{A} = \mathcal{A}\mathcal{T}$.

Since A and B are hyponormal on H it implies $\mathcal{A}$ is also hyponormal on $H \oplus H$.

With $\mathcal{A}\mathcal{T} = \mathcal{T}\mathcal{A}$ we have $\mathcal{B}\mathcal{T} = \mathcal{T}\mathcal{B} = \mathcal{T}\mathcal{A}$ $\Longrightarrow$ $\mathcal{B}\mathcal{T} = \mathcal{T}\mathcal{A}$ where $\mathcal{B}$ is the polar decomposition os B on $H \oplus H$.

Thus the equation;

$\mathcal{A}^n\mathcal{X} - \mathcal{X}\mathcal{B}^n + \sum_{i=1}^{n}\mathcal{A}^{n-i-1}(\mathcal{T} - (\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{B}))\mathcal{A}^i = n\mathcal{A}^{n-1}\mathcal{T}$ holds.

By applying the equation and the workings of the previous Theorem 3.1 we have the result.

Lemma 3.4 Let $A \in F_H$ (H), then the following are equivalent;

(i) $I \in \overline{R(\delta_A)}$

(ii) There exist $T \in \{A\}^\#$ such that $T \in \overline{R(\delta_A)}$

(iii) $\overline{R(\delta_A)}$ contains all positive invertible hyponormal operators in $\{A\}^\#$

(iv) $\overline{R(\delta_A)} = F_H$ (H).

Proof. i) $\Longrightarrow$ ii) Suppose $I \in \overline{R(\delta_A)}$ then also $I \in \{A\}^\#$ implying existence of an invertible operator $T \in F_H$ (H) such that $TT^{-1} = T^{-1}T = I \in \{A\}^\#$.

Then we have a polynomial P of degree n such that $P^k(A)X_n - X_nP^k(A) \longrightarrow P^{k+1}(A)I$ where P^k is the k^{th} derivative of P and (X_n) is a sequence of operators of F_H (H) [24]

$\Rightarrow P^k(A)X_n - X_nP^k(A) \rightarrow P^{k+1}(A)TT^{-1}$

Multiplying each term from the right by T we have

$P^k(A)X_nT - X_nP^k(A)T \rightarrow P^{k+1}(A)TT^{-1}T$

By polynomial properties we have, $P(A)X_nT - X_nP(A)T \rightarrow P^1(A)T$.

$\Rightarrow P(A)X_nT - TX_nP(A) \rightarrow P^1(A)T$

$\Rightarrow T \in \{A\}^\#$.

Also $I \in \overline{R(\delta_A)}$ implying existence of a sequence of operators (X_n) such that $AX_n - X_nA \rightarrow I$ and since A is finite hyponormal operator we have $\|AX_n - X_nA - I\| \geq \|I\|$ implying existence of an invertible operator $T \in F_H$ (H) such that $\|AX_n - X_nA - TT^{-1}\| \geq \|TT^{-1}\|$

$\Rightarrow \|AX_n - X_nA - T\|\|T^{-1}\| \geq \|T\|\|T^{-1}\|$

$\Rightarrow \|AX_n - X_nA - T\| \geq \|T\|$

$\Rightarrow T \in \overline{R(\delta_A)}$.

ii) $\Longrightarrow$ i) Suppose there exist an operator P such that $P \in \overline{R(\delta_A)} \cap \{A\}^\#$.

Then there exist a sequence of operators $\{X_n\}$ of $F_H^{(H)}$ such that $\|P - (AX_n - X_nA)\| \longrightarrow 0$ as setting $n \longrightarrow \infty$.

Setting $T_n = P^{-1}X_n$ we have; $\|P^{-1}P - P^{-1}(AX_n - X_nA)\| = \|I - (AP^{-1}X_n - P^{-1}X_nA)\| = \|I - (AT_n - T_nA)\|$ and since $P \in \{A\}^\#$ implies that $P^{-1} \in \{A\}^\#$ we have; $\|I - (AT_n - T_nA)\| = \|I - (P^{-1}AX_n - P^{-1}X_nA)\|$

$= \|P^{-1}(P - (AX_n - X_nA))\| \leq \|P^{-1}\|\|P - (AX_n - X_nA)\|$.

Since $\|P - (AX_n - X_nA)\| \rightarrow 0$ as $n \rightarrow \infty$ it follows that $\|I - (AT_n - T_nA)\| \rightarrow 0$ as $n \longrightarrow \infty$ and

hence $I \in \overline{R(\delta_A)}$.

i) $\Longrightarrow$ iii) If $I \in \overline{R(\delta_A)}$, then there exist a sequence (X_n) of operators of F_H (H) such that $\|I - (AX_n - X_nA)\| \rightarrow 0$ as $n \longrightarrow \infty$ and also since $I \in \overline{R(\delta_A)}$ then for every invertible operator $B \in \overline{R(\delta_A)}$ there exists $B^{-1} \in \overline{R(\delta_A)}$ such that $I = BB^{-1}$ and hence

$\|I - (AX_n - X_nA)\| = \|BB^{-1} - (AX_n - X_nA)\| \leq \|B^{-1}\|\|B - (AX_n - X_nA)\|$

$\Longrightarrow \|B - (AX_n - X_nA)\| \longrightarrow 0$ as $n \longrightarrow \infty$ which implies that $(AX_n - X_nA) \longrightarrow B \in \overline{R(\delta_A)}$, by definition and by setting $(X_n) \longrightarrow B$ as $n \longrightarrow \infty$ we have $BA = AB$ for an arbitrary positive invertible hyponormal operator $B \in \{A\}^\#$.

iii) $\Longrightarrow$ iv) Let $B \in F_H$ (H) then by definition $\{A\}^\# = \{B \in F_H(H) : AB = BA\}$ for all $A \in F_H$ (H). Implying that $F_H(H) \subset \{A\}^\#$ then by iii) we have $F_H(H) \subset \overline{R(\delta_A)}$ and hence $A \in F_H(H) \subset \overline{R(\delta_A)}$. On the other hand, let $X \in \overline{R(\delta_A)}$ then we need to show that $X \in F_H$ (H). Let $B \in \{A\}^\#$ be a positive invertible hyponormal operators such that by iii) we have $B \in \overline{R(\delta_A)}$ then there exist a sequence (X_n) such that $AX_n - X_nA \longrightarrow B$ $\Longrightarrow \|B - (AX_n - X_nA)\| \longrightarrow 0$ as $n \longrightarrow \infty$ and by the vanishing properties of all operators in $\overline{R(\delta_A)}$ we have $AX_n = X_nA$ as $n \longrightarrow \infty$ and by setting $(X_n) \longrightarrow X$ as $n \longrightarrow \infty$ then $AX_n = X_nA$ becomes $AX = XA$ implying that $X \in F_H$ (H) hence $\overline{R(\delta_A)} = F_H$ (H).

Theorem 3.5 Let $A, B \in F_H^{(H)}$ be finite rank hyponormal operators such that $\|A^n\| = \|A\|^n \leq 1$ where $n \in \mathbb{N}$. Suppose $I \in \overline{R(\delta_{A,B})}$ then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.

Proof. Since $I \in \overline{R(\delta_{A,B})}$ then by Lemma 3.4 there exist $T \in F_H$ (H) such that $AT = TB$ and hence the equation;

$A^nX - XB^n + \sum_{i=1}^{n} A^{n-i-1}(T - (AX - XB))A^i = nA^{n-1}T$ holds.
Following the workings of Theorem 3.2 we have the result as follows;
For n = 1, we have AX – XB + T – (AX – XB) = T
$\Rightarrow \|AX - XB\| + \|T - (AX - XB)\| \geq \|T\|$
For n = 2 we have $A^2X - XB^2 + 2(T - (AX - XB))A = 2AT$
$\Rightarrow \|A^2X - XB^2\| + 2\|T - (AX - XB)\| \geq 2\|T\|$
$\Rightarrow \frac{\|A^2X - XB^2\|}{2} + \|T - (AX - XB)\| \geq \|T\|$
Similarly for n = 3 we have; $\|A^3X - XB^3\| + 3\|T - (AX - XB)\| \geq 3\|T\|$
$\frac{\|A^3X - XB^3\|}{3} + \|T - (AX - XB)\| \geq \|T\|$
For an arbitrary n ϵ ℕ we have; $\frac{\|A^nX - XB^n\|}{n} + \|T - (AX - XB)\| \geq \|T\|$
Taking $n \to \infty$ we have; $\|T - (AX - XB)\| \geq \|T\|$
Theorem 3.6 ([30], Theorem 5) Let $T \in F_H^{(H)}$ be hyponormal operator such that $T^n = N$ where n is a positive integer. If N is a normal operator, then T is also normal.
Theorem 3.7 Let A, B ϵ F_H (H) be hyponormal operators. Suppose there exist a unitary operator U such that $A^n = U$ and $B^n = U$ where n and m are distinct positive integers. Then for every $X \in F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.
Proof. With $A^n = U$ and $B^n = U$ implies A^n and B^n are similar to unitary operators [25]. Thus (A, B) reduces to a pair of normal operators. This is also true by the fact that A and B are hyponormal operators such that $A^n = U$ and $B^n = U$ implying that A and B are normal operators by Theorem 3.6.
Let A == $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be the matrix representation of A relative to the orthogonal decomposition $H = H_1 = \overline{R(T)} \oplus \overline{R(T)}^{\perp}$ and B == $\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ be the matrix representation of B relative to the orthogonal decomposition $H = H_2 = \ker(T)^{\perp} \oplus \ker(T)$.
Now taking operators $T, X : H_1 \longrightarrow H_2$ having matrix representations;
$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ and $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$.
Then AX – XB – T = $\begin{pmatrix} A_1X_1 - X_1B_1 - T_1 & A_1X_2 - X_2B_2 \\ A_2X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{pmatrix}$
Since the norm of an operator matrix supersedes the norm of its diagonal entry we have;
$\|(AX - XB) - T\| \geq \|(A_1X_1 - X_1B_1) - T_1\| \geq \|T_1\| \geq \|T\|$
$\Rightarrow \|(AX - XB) - T\| \geq \|T\|$.
Theorem 3.8 Let A, B ϵ F_H (H). If A is hyponormal B^* is m-hyponormal, then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.
Proof. Yoshino [32] shows that if A is hyponormal and B^* is m-hyponormal, then AT = TB $\Rightarrow A^*T = TB^*$ i.e the pair (A, B) has Fuglede-Putnam (FP) property. Yusun [33, Theorem 1] shows that if (A, B) is (FP) pair then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.
Theorem 3.9 Let A, B ϵ F_H (H) be hyponormal operators. If $T \in F_H^{(H)}$ is generalized nilpotent hyponormal operator, then for all $X \in F_H^{(H)}$ we have $\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.
Proof. With T being a generalized nilpotent ($\lim_{n \to \infty} \|T^n\|^{1/n} = 0$) hyponormal operator, then its norm is necessarily zero [7], then by basic properties of operator norm we have;
$\|T - (AX - XB)\| \geq \|T\|$ for all $T \in \ker\delta_{A,B}$.

Orthogonality of finite rank derivations

Here we establish range-kernel orthogonality of finite rank derivations implemented by hyponormal operators.
Theorem 3.10 ([25], Theorem 2.7.1) Let A, B ϵ B(H). If every positive operator in $\overline{R(\delta_A)}$ vanish, then $\overline{R(\delta_{A,B})} \cap \ker\delta_{A^*,B^*} = \{0\}$, for every operator B ϵ B(H).
Theorem 3.11 Let A ϵ F_H (H) be a nilpotent hyponormal operator of index two and if $I \in \overline{R(\delta_A)}$, then $\overline{R(\delta_{A,B})} \cap \ker\delta_{A^*,B^*} = \{0\}$, for every operator B ϵ F_H (H).
Proof. Since A ϵ F_H (H) and $I \in \overline{R(\delta_A)}$ the by Lemma 3.4 there exist a positive operator $T \in \overline{R(\delta_A)}$ such that for a sequence (X_n) of operators in F_H (H) we have;
$AX_n - X_nA \longrightarrow T$ (1)
Hence $A^2X_n - X_nA^2 \longrightarrow AT + TA$ this follows from [25].
Since A is nilpotent of index two we have;
$0 = A^2X_n - X_nA^2 \longrightarrow AT + TA \Rightarrow AT + TA = 0$ and so we have AT = TA = 0.

Applying T from the right and left on (1) we obtain $TAX_nT - TX_nAT \longrightarrow T^3$ which implies that $T^3 = 0$ thus $T = 0$.

We have shown that $T \in \overline{R(\delta_A)}$ vanishes and thus by Theorem 3.10 above we have the result.

Theorem 3.11 Let A, B $\in F_H$ (H) be finite rank hyponormal operators such that A and B^* are isometric. Then $\overline{R(\delta_{A,B})} \cap \ker\delta_{A,B} = \{0\}$ vanish.

Proof. Since $\overline{R(\delta_{A,B})} \cap \ker\delta_{A,B} = \{0\}$ (1)

Then there exist a sequence (X_n) of operators such that $AX_n - X_nB \longrightarrow T$

Multiply each term to the right by T^* to obtain;

$AX_nT^* - X_nBT^* \longrightarrow TT^*$ (2)

Also from (1) we have $AT = TB$ (3)

Multiply each term to the left by A^* and to the right by B^* to get,

$A^*ATB^* = A^*TBB^*$

$\Longrightarrow TB^* = A^*T$

Taking adjoint we have; $BT^* = T^*A$ (4)

Applying (4) to (2) we obtain $A(X_nT^*) - (X_nT^*)A \longrightarrow TT^* \Longrightarrow TT^* \in \overline{R(\delta_A)}$ (5)

From the left we apply T both sides to line (4) $TBT^* = (TT^*)A$ (6)

From the right we apply T^* both sides to line (3) $A(TT^*) = TBT^*$ (7)

Comparing (6) and (7) we have $(TT^*)A = A(TT^*) \Longrightarrow TT^* \in \{A\}^\#$ (8)

By (5) and (8) we have $TT^* \in \overline{R(\delta_A)} \cap \{A\}^\# = \{0\}$

Therefore $TT^* = 0$ implying $T = 0$.

Theorem 3.11 Let A $\in F_H$ (H) and B $\in F_H$ (H) be similar. Suppose $A^* \in \overline{R(\delta_{A^*})} \cap \{A^*\}^\# = \{0\}$ then $B^* \in \overline{R(\delta_{B^*})} \cap \{B^*\}^\# = \{0\}$.

Proof. A and B are similar then by Nzimbi [27, Theorem 2.11] A^* and B^* are also similar and hence by [67] and hence by Mohamed [26, Theorem 2.4] there exist an invertible operator S $\in F_H$ (H) such that $B^* = S^{-1}A^*S$.

Then for all X $\in F_H$ (H) we have $S^{-1}(A^*X - XA^*)S = B^*(S^{-1}XS) - (S^{-1}XS)B^*$

Thus $S^{-1}\overline{R(\delta_{A^*})}S = \overline{R(\delta_{B^*})}$

Hence $\overline{R(\delta_{B^*})} \cap \{B^*\}^\# = \{S^{-1}\overline{R(\delta_{A^*})}S\} \cap \{S^{-1}\{A^*\}^\# S\}$

$\Longrightarrow \overline{R(\delta_{B^*})} \cap \{B^*\}^\# = \{S^{-1}\overline{R(\delta_{A^*})} \cap \{A^*\}^\# S\} = \{0\}$

Theorem 3.11 Let A, B $\in F_H$ (H) be such that A is similar to B. Suppose that $\overline{R(\delta_{P(A)})} \oplus \{P(A)\}^\# = \{0\}$ for some polynomial P. Then the set $T = \overline{R(\delta_B)} \oplus \{B\}^\#$ is nilpotent.

Proof. Let P be a polynomial of degree n for which P^k denotes the k^{th} derivative of P and suppose $T = \overline{R(\delta_B)} \oplus \{B\}^\#$.

Then there exist a sequence of operators (X_n) in F_H (H) such that $BX_n - X_nB \longrightarrow T \in \{B\}^\#$

Then by Mecheri [19, Theorem 2] we have; $P(B)X_n - X_nP(B) \longrightarrow P^1(B)T$

But A is similar to B implying existence of an operator S $\in F_H$ (H) such that $B = S^{-1}AS$

Then $P(S^{-1}AS)X_n - X_nP(S^{-1}AS) \longrightarrow P^1(S^{-1}AS)T$

By polynomial properties we have; $P(A)X_n - X_nP(A) \longrightarrow P^1(A)T$

This shows that $P^1(A)T \in \overline{R(\delta_{P(A)})} \oplus \{P(A)\}^\# = \{0\}$

Also $P^1(B)X_n - X_nP^1(B) \longrightarrow P^2(B)T$

$\Longrightarrow P^1(S^{-1}AS)X_n - X_nP^1((S^{-1}AS) \longrightarrow P^2(S^{-1}AS)T$

$\Longrightarrow P^1(A)X_n - X_nP^1(A) \longrightarrow P^2(A)T$

$\Longrightarrow T P^1(A)X_nT - TX_nP^1(A)T \longrightarrow P^2(A)T^3$

Repeating the process we have $T^k = 0$ hence T is nilpotent.

Theorem 3.11 Let A $\in F_H$ (H) be k-quasihyponormal and $B^* \in F_H$ (H) be injective p-hyponormal operator, then $\overline{R(\delta_{A,B})} \cap \ker\delta_{A,B} = \{0\}$.

Proof. According to Bachir [4, Theorem 3.3] if A $\in F_H$ (H) is k-quasihyponormal and $B^* \in F_H$(H) is injective p-hyponormal operator then the pair (A, B) has the (FP) property. But A $\in F_H$(H) being k-quasihyponormal implies A is hyponormal and B^* $\in F_H$ (H) being injective p-hyponormal implies B is also hyponormal. Hence (A, B) is a pair of hyponormal operators with Fuglede-Putnam (FP) property such that B is injective and then by Mohamed [26, Lemma 3.3] we have $\overline{R(\delta_{A,B})} \cap \ker\delta_{A,B} = \{0\}$.

Conclusions

In the present work, authors have studied and established range-kernel orthogonality inequality for finite rank generalized derivations implemented by hyponormal operators. Here orthogonality is in Birkhoff sense defined on generalized derivation $\delta_{A,B}: F_H$ (H) $\rightarrow F_H$ (H) defined as $\delta_{A,B}(X) = AX - XB$. Considering the same sense of orthogonality, it would be interesting to establish range-kernel orthogonality inequality for hyponormal operators for the adjoint of generalized derivation; $\delta^*_{A,B}: F_H$ (H) $\rightarrow F_H$ (H) defined as

$\delta^*_{A,B}(X) = A^*X - XB^*$. Since there are different kinds of orthogonality such as Pythagorean, Isosceles, Roberts, Singer, James orthogonality, it would be interesting to establish range-kernel orthogonality for hyponormal operators using any other kind of orthogonality in the Hilbert space. In fuctional analysis, like many fields of science, mathematics and technology, orthogonality is central. In operator theory, Birkhoff orthogonality and semi-inner product have been used to characterize best approximation and existence of best approximation and best co-approximation in a normed space. Also in normed spaces, studies show that Birkhoff orthogonality implies best approximation and best approximation implies Birkhoff orthogonality. By polar decomposition and φ-Gateaux derivative of the norm, it has been established orthogonal operators in C_1-classes and further established best approximation in a complex Banach space. All these can be considered for other forms of orthogonality. With regard to von Neumann Schatten p-class, mathematicians have used polar decomposition of T, i.e. $T = U|T|$ to establish range-kernel orthogonality for normal operators with (FP) property. It would be of interest to establish operators with (FP) property for subnormal, m-hyponormal and dominant operators and further establish orthogonality for Schatten p-class and compact operators.

Conflict of interest

Authors declare no conflict of interest.

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