The Line

Find the equations of the two lines $\boxed{\text{parallel}}$ to $4x - 3y + 8 = 0$ if the perpendicular distance from the origin to each line is 4.

$(0,0)$ $d=4$

$4x - 3y + c = 0$ $(0,0)$ $d=4$

$a=4$ $b=-3$ $c=c$ x, y

$$\frac{|(4)(0)+(-3)(0)+c|}{\sqrt{(4)^2+(-3)^2}} = 4$$

$$\frac{|c|}{\sqrt{25}} = 4$$

$$\frac{|c|}{5} = 4$$

$$(5)\left(\frac{|c|}{5}\right) = (4)(5)$$

$|c| = 20$

$c = 20$ or $c = -20$

L_0: $4x - 3y + 20 = 0$

or $4x - 3y - 20 = 0$.

(i) Calculate the perpendicular distance from the point $(-1, -5)$ to the line $3x - 4y - 2 = 0$.

(ii) The point $(-1, -5)$ is equidistant from the lines $3x - 4y - 2 = 0$ and $3x - 4y + k = 0$, where $k \neq -2$. Find the value of k.

i) $(-1,-5)$ $3x - 4y - 2 = 0$

x, y $a=3$ $b=-4$ $c=-2$

$$d = \frac{|(3)(-1)+(-4)(-5)+(-2)|}{\sqrt{(3)^2+(-4)^2}}$$

$$d = \frac{|-3+20-2|}{\sqrt{25}} = \frac{15}{5} = 3$$

(ii) $3x - 4y + k = 0$ $d=3$ $(-1,-5)$

$a=3$ $b=-4$ $c=k$ x, y

$$3 = \frac{|(3)(-1)+(-4)(-5)+(k)|}{\sqrt{(3)^2+(-4)^2}}$$

$$3 = \frac{|17+k|}{5}$$

$$(5)(3) = \left(\frac{|17+k|}{5}\right)(5)$$

$15 = |17+k|$

$17 + k = 15$

$-17 \quad -17$

$k = -2$

or

$17 + k = -15$

$-17 \quad -17$

$k = -32$

The straight lines $y = k^2x + 12$ and $2ky = 4x + 5$ are perpendicular, $k \neq 0$.

(i) Find the value of k.

(ii) Find the point of intersection of the two lines. Simultaneous equations!

$y = k^2x + 12$

slope: k^2

$2ky = 4x + 5$

$\frac{2ky}{2k} = \frac{4x}{2k} + \frac{5}{2k}$

$y = \frac{4}{2k}x + \frac{5}{2k}$

slope: $\frac{4}{2k}$.

then $m_1 \times m_2 = -1$ b/o

$\frac{k^2}{1} \times \frac{4}{2k} = -1$

simplify $\frac{4k^2}{2k} = -1$

$2k = -1$

$k = -\frac{1}{2}$

(ii) $y = (-\frac{1}{2})^2x + 12$

and $2(-\frac{1}{2})y = 4x + 5$

$y = \frac{1}{4}x + 12$

(x4)

$4y = x + 48$

$-y = 4x + 5$

Rearrange:

$4y - x = 48 \; (\times -4)$

$-y - 4x = 5$

$-16y + 4x = -192$

$-y + 4x = 5$

$-17y = -187$

$y = 11$

$y = 11$, $4y = x + 48$

$4(11) = x + 48$

$44 = x + 48$

$-48 \quad -48$

$-4 = x$

ℓ is the line $4x + 3y - 5 = 0$.

Verify that $(2, -1) \in \ell$.

x y

$4x + 3y - 5 = 0$

$4(2) + 3(-1) - 5 = 0$

$8 - 3 - 5 = 0$

$0 = 0$

$\therefore (-2, -1) \in \ell$ ✓.

Find the equations of the lines through the point (2, 4) which make angles of 45° with the line $x - 2y - 6 = 0$.

$\tan\theta = \pm\frac{M_1 - M_2}{1 + M_1 M_2}$

$-2y = -x + 6$

$\frac{-2y}{-2} = \frac{-x}{-2} + \frac{6}{-2}$

$y = \frac{1}{2}x - 3$

$M = \frac{1}{2}$

Let other slope be 's'

$\tan 45° = \pm\frac{(\frac{1}{2}) - s}{1 + (\frac{1}{2})(s)}$

$1 = \pm\left(\frac{\frac{1}{2} - s}{1 + \frac{1}{2}s}\right)$

$(1 + \frac{1}{2}s)1 = \pm\left(\frac{\frac{1}{2} - s}{\cancel{1 + \frac{1}{2}s}}\right)\cancel{(1 + \frac{1}{2}s)}$

$1 + \frac{1}{2}s = \pm(\frac{1}{2} - s)$

So: $1 + \frac{1}{2}s = +(\frac{1}{2} - s)$ or $1 + \frac{1}{2}s = -(\frac{1}{2} - s)$

$1 + \frac{1}{2}s = \frac{1}{2} - s$ $\qquad$ $1 + \frac{1}{2}s = -\frac{1}{2} + s$

$2 + \cancel{s} = 1 - 2s$ $\qquad$ $2 + \cancel{s} = -1 + 2s$

$-s \quad -s$ $\qquad$ $-s \quad -s$

$2 = 1 - 3s$ or $2 = -1 + s$

$-1 \quad -1$ $\qquad$ $+1 \quad +1$

$1 = -3s$ $\qquad$ $\boxed{3 = s}$

$\boxed{-\frac{1}{3} = s}$

$y - y_1 = M(x - x_1)$ $(2,4)$ $s = -\frac{1}{3}$

$y - 4 = -\frac{1}{3}(x - 2)$

$3(y - 4) = -1(x - 2)$

$3y - 12 = -x + 2$

$\boxed{x + 3y - 14 = 0}$

or $\qquad$ $(2,4)$ $s = 3$

$y - 4 = 3(x - 2)$

$y - 4 = 3x - 6$

$\boxed{y - 3x + 2 = 0}$

(i) A(−7, 3) and B(8, −2) are two points.
Find the coordinates of the point that divides [AB] in the ratio 2:3.

(ii) ℓ is the line $2x + ky = 6$.

(a) Find, in terms of k, the points where ℓ intersects the x-axis and y-axis. $\quad y = 0 \quad x = 0$!

(b) If the area of the triangle formed by ℓ, the x-axis and the y-axis is k square units, find the value of k.

i) $A(-7, 3)$ $B(8, -2)$ $\qquad 2:3$

$\quad x_1 y_1 \qquad x_2 y_2 \qquad a:b$

$\left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$

$\left(\frac{(3)(-7) + (2)(8)}{(3) + (2)}, \frac{(3)(3) + (2)(-2)}{(3) + (2)}\right)$

$\left(\frac{-21 + 16}{5}, \frac{9 - 4}{5}\right)$

$(-1, 1)$

(ii)
a) Cuts x-axis: $2x + k(0) = 6$

$2x = 6$

$x = 3$ $\quad (3, 0)$

Cuts y-axis $2(0) + ky = 6$

$ky = 6$

$y = \frac{6}{k}$ $\quad (0, \frac{6}{k})$

ii) b) $A = \frac{1}{2}|x_1y_2 - x_2y_1|$ $\quad (3, 0)$ $(0, \frac{6}{k})$ $A = k$

$\qquad x_1 y_1 \qquad x_2 y_2$

$k = \frac{1}{2}\left|(3)(\frac{6}{k}) - (0)(0)\right|$

$k = \frac{1}{2}\left|\frac{18}{k}\right|$

$2k = \left|\frac{18}{k}\right|$

So $2k = \frac{18}{k}$ or $-2k = \frac{18}{k}$

$2k^2 = 18$ $\qquad$ $-2k^2 = 18$

$k^2 = 9$ $\qquad$ $k^2 = -9$

$k = \pm 3$ $\qquad$ $k = \sqrt{-9}$ N/A.

The equation ~~of the line~~ ℓ is $3x - 2y + 6 = 0$.
Find the equation of the line perpendicular to ℓ that contains the point $(-1, 4)$.

$-2y = -3x - 6$

$\frac{\cancel{-2}y}{\cancel{-2}} = \frac{-3x}{-2} - \frac{6}{-2}$

$y = \frac{3}{2}x + 3$

$m = \frac{3}{2}$

flip & change sign

$\frac{-2}{3}$

x, y

$y - y_1 = m(x - x_1)$

$y - 4 = -\frac{2}{3}(x - (-1))$

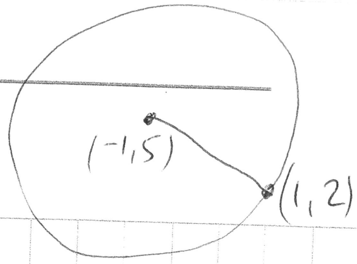
$3(y-4) = \cancel{3}\left(-\frac{2}{\cancel{3}}(x+1)\right)$

$3y - 12 = -2x - 2$

$2x + 3y - 10 = 0$

The Circle

A circle has centre $(-1, 5)$ and passes through the point $(1, 2)$.
(i) Find the the length of the radius of the circle.
(ii) Write down the equation of the circle.



(i) $(-1,5)$ $(1,2)$
$x_1 y_1$ $x_2 y_2$

$\sqrt{(1--1)^2+(2-5)^2}$

$\sqrt{(2)^2+(-3)^2}$

$\sqrt{13}$

(ii) $(x-h)^2+(y-k)^2=r^2$

$(x-(-1))^2+(y-5)^2=(\sqrt{13})^2$

$(x+1)^2+(y-5)^2=13$

Find the coordinates of the centre and radius length of the circle $x^2 + y^2 - 2x - 4y - 9 = 0$.
Hence write down the equation of the circle with the origin as centre and which has the same radius length as the given circle.

$2gx=-2x$
$2g=-2$
$g=-1$

$2fy=-4y$
$2f=-4$
$f=-2$

radius: $\sqrt{g^2+f^2-c}$

$r=\sqrt{(-1)^2+(-2)^2-(-9)}$

$r=\sqrt{1+4+9}$

$r=\sqrt{14}$

Centre $(1,2)$
$(-g,-f)$

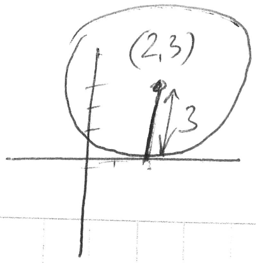
$(0,0)$
h,k

$(x-h)^2+(y-k)^2=r^2$

$(x-0)^2+(y-0)^2=(\sqrt{14})$

$x^2+y^2=14$

Find the equation of the circle with centre (2, 3) and which touches the x-axis.



$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x-2)^2 + (y-3)^2 = (3)^2$$
$$(x-2)^2 + (y-3)^2 = 9$$

Show that the line $3x - 4y + 25 = 0$ is a tangent to the circle $x^2 + y^2 = 25$.

$3x - 4y + 25 = 0$

$a = 3 \quad b = -4 \quad c = 25$

$d = 5 \quad \underset{x_1\ y_1}{(0, 0)}$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$5 = \frac{|(3)(0) + (-4)(0) + (25)|}{\sqrt{(3)^2 + (-4)^2}}$$

$$5 = \frac{|25|}{\sqrt{25}}$$

$(0,0)$

$r = \sqrt{25} = 5$

$$5 = \frac{25}{5}$$

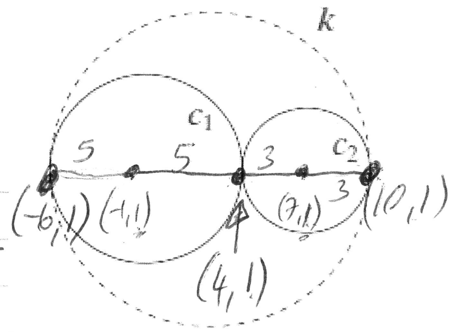
$5 = 5$ ✓

$\therefore$ it is a tangent.

c_1: $x^2 + y^2 + 2x - 2y - 23 = 0$ and
c_2: $x^2 + y^2 - 14x - 2y + 41 = 0$ are two circles.

(i) Prove that c_1 and c_2 touch externally.

(ii) k is a third circle.
Both c_1 and c_2 touch k internally.
Find the equation of k.



C_1 $\quad 2gx = 2x \quad 2fy = 2y \quad R_1 = \sqrt{g^2 + f^2 - c}$

$2g = 2 \quad 2f = -2 \quad = \sqrt{(1)^2 + (-1)^2 - (-23)}$

$g = 1 \quad f = -1 \quad = \sqrt{25} = 5$

Centre
$(-g, -f) : (-1, 1)$

C_2 $\quad 2gx = -14x \quad 2fy = -2y \quad r_2 = \sqrt{(-7)^2 + (-1)^2 - (41)}$

$2g = -14 \quad 2f = -2 \quad \sqrt{9} = 3$

$g = -7 \quad f = -1$

Centre $(7, 1)$

$(-1, 1)$ $\quad (7, 1)$
$x_1\ y_1 \quad x_2\ y_2$

Distance between centres $\sqrt{(7 - -1)^2 + (1 - 1)^2} = \sqrt{64} = 8$

$8 = 5 + 3$ ✓ They touch externally.

Diameter $= 16$
Radius of $k = 8$

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

$\left(\frac{-6+10}{2}, \frac{1+1}{2}\right)$

$(2, 1)$

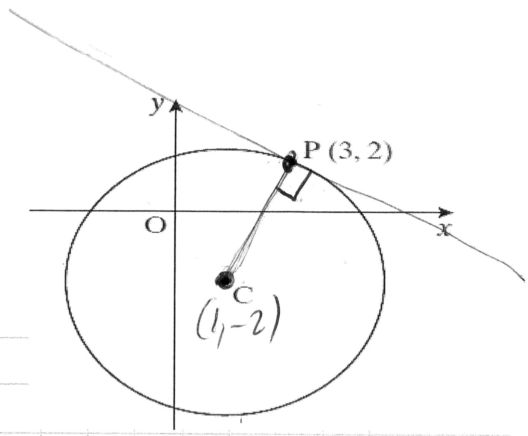
Circle k: $(x-2)^2 + (y-1)^2 = 64$

The equation of the given circle is
$$(x - 1)^2 + (y + 2)^2 = 20$$

(i) Verify that the point P(3, 2) is on the circle.

(ii) Write down the coordinates of C, the centre of the circle.

(iii) Hence find the equation of the tangent to the circle at the point P.



(i) $\begin{matrix}(3, 2)\\ x, y\end{matrix}$ $\quad ((3)-1)^2 + ((2)+2)^2 = 20$

$4 + 16 = 20$ ✓ It is on the circle

(ii) $(1, -2)$ centre

(iii) $y - y_1 = m(x - x_1)$ $\quad \begin{matrix}(3, 2)\\ x_1 y_1\end{matrix}$

Slope of radius: $\frac{y_2 - y_1}{x_2 - x_1} \Rightarrow \frac{2 - -2}{3 - 1} = \frac{4}{2} = 2$

So slope of tangent $= -\frac{1}{2}$

$y - 2 = -\frac{1}{2}(x - 3)$

$2(y-2) = 2\left(-\frac{1}{2}(x-3)\right)$

$2(y-2) = -1(x-3)$

$2y - 4 = -x + 3$

$2y + x - 7 = 0$

Points (2, 5) and (−2, 1) lie on the circle $x^2 + y^2 + 2gx + 2fy + 7 = 0$.
(i) Make two equations in g and f.
(ii) Solve this pair of equations to find the values of g and f.
(iii) Hence find the equation of the circle, and give its centre and radius.

$(2)^2+(5)^2+2g(2)+2f(5)+7=0$
$4+25+4g+10f+7=0$
$4g+10f=-36$

$(-2)^2+(1)^2+2g(-2)+2f(1)+7=0$
$4+1-4g+2f+7=0$
$-4g+2f=-12$

$\cancel{4g}+10f=-36$
$\cancel{-4g}+2f=-12$
$12f=-48$
$f=-4$

$f=-4,\ 4g+10(-4)=-36$
$4g=-36+40$
$4g=4$
$g=1$

$x^2+y^2+2(1)x+2(-4)y+7=0$
$x^2+y^2+2x-8y+7=0$

Centre $(-g,-f)$
$(-1,4)$
Radius: $\sqrt{(+1)^2+(-4)^2-(7)}=\sqrt{10}$.

Find the two values of k for which $8x + 3y + k = 0$ is a tangent to the circle $x^2 + y^2 + 4x - 3y - 12 = 0$.

$2gx=4x$ $\quad 2fy=-3y$
$2g=4$ $\quad 2f=-3$
$g=2$ $\quad f=-3/2$

radius: $\sqrt{g^2+f^2-c}$
$r=\sqrt{(2)^2+(-3/2)^2-(-12)}$
$r=\sqrt{18.25}=\frac{\sqrt{73}}{2}$

Centre $(-2, 3/2)$
$(-g,-f)$

$d=\sqrt{73}/2$ $\quad (-2, 3/2)$ $\quad 8x+3y+k=0$
x_1, y_1 $\quad a=8\ b=3\ c=k$

$$\frac{\sqrt{73}}{2}=\frac{|(8)(-2)+(3)(3/2)+k|}{\sqrt{(8)^2+(3)^2}}$$

$$\frac{\sqrt{73}}{2}=\frac{|-11\tfrac{1}{2}+k|}{\sqrt{73}}$$

$$\left(\frac{\sqrt{73}}{2}\right)(\sqrt{73})=\left(\frac{|-11\tfrac{1}{2}+k|}{\cancel{\sqrt{73}}}\right)(\cancel{\sqrt{73}})$$

$$\frac{73}{2}=|-11\tfrac{1}{2}+k|$$

So: $73/2=-11\tfrac{1}{2}+k$ or $-73/2=-11\tfrac{1}{2}+k$
$+11\tfrac{1}{2}$ $\quad +11\tfrac{1}{2}$ $\qquad +11\tfrac{1}{2}$ $\quad +11\tfrac{1}{2}$
$48=k$ $\qquad -25=k$

Probability

Karen has two 5c and four 10c coins in her purse. At random, she takes out one coin and then a second coin (without replacing the first). Find the probability that

(i) the first coin is a 10c and the second coin a 5c
(ii) the two coins are worth 15c
(iii) the two coins are worth 20c.

(i) $\frac{4}{6} \times \frac{2}{5} = \frac{8}{30} = \frac{4}{15}$

(ii) 10c and 5c or 5c and 10c

$\left(\frac{4}{6} \times \frac{2}{5}\right) + \left(\frac{2}{6} \times \frac{4}{5}\right) = \frac{16}{30} = \frac{8}{15}$.

(iii) 10c and 10c

$\frac{4}{6} \times \frac{3}{5} = \frac{2}{5}$.

A bag contains 5 red and 4 green discs, identical in all but colour. Three discs are drawn at random from the bag without replacement. Find the probability that

(i) they are all the same colour
(ii) at least one is red

(i) R and R and R or G and G and G

$\left(\frac{5}{9} \times \frac{4}{8} \times \frac{3}{7}\right) + \left(\frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}\right) = \frac{1}{6}$.

(ii) $1 - P(\text{no red})$

$1 - \left(\begin{array}{c}\text{No red and No red and No red} \\ \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7}\end{array}\right) = \frac{20}{21}$.

(i) How many arrangements can be made with the letters of the word SOLDIER if all the letters are taken at a time?

(ii) How many of these arrangements begin with the letters SO in that order?

(iii) How many of the arrangements begin and end with a consonant?

(i) $7\times6\times5\times4\times3\times2 = 7! = 5040$

(ii) $\underline{1}\times\underline{1}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1} = 120$

(iii) $\underline{4}\times\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times\underline{3} = 1440$

Extra What if all the vowels must be together?

$\boxed{O\,I\,E}\ \boxed{S}\boxed{L}\boxed{D}\boxed{R}$

$\underline{5}\times\underline{4}\times\underline{3}\times\underline{2}\times\underline{1}\times 3! = 720$

Extra What if the three vowels cannot all be together?

$5040 - 720 = 4320$

(i) Give an equation involving probabilities which represents the statement 'the events L and M are mutually exclusive'. Explain what is meant by 'mutually exclusive events'.

(ii) Janelle's mathematics class has 22 students.
Four students are selected at random to enter a mathematics competition.
(a) In how many ways can the four students be selected?
(b) In how many of the selections is Janelle included?
(c) Now find the probability that Janelle is included to enter the competition.

(i) Events L and M cannot happen at the same time.

(ii) a) ${}^{22}C_4 = \binom{22}{4} = 7315$

b) ${}^{21}C_3 = 1330$

c) $\frac{1330}{7315} = \frac{2}{11}$.

A local football team wins 80% of its home matches. Find the probability that (i) the first win occurs in the 4th match.
(ii) the first loss occurs in the 4th match.

(i) Consider: No wins in first 3 matches and win in 4th match

Prob. win: 0.8

Prob. lose: 0.2

$$\binom{3}{0}(0.8)^0(0.2)^3 \times 0.8 = \frac{4}{625}$$

(ii) Consider: 3 wins in first 3 matches and lose in 4th match

$$\binom{3}{3}(0.8)^3(0.2)^0 \times 0.2 = \frac{64}{625}$$

⇒ Extra: what about getting at least one loss in 4 matches

$1 - P(\text{no losses})$

$$1 - \left[\binom{4}{0}(0.8)^0(0.2)^4\right] = \frac{624}{625}$$

20% of the items produced by a machine are defective. Four items are chosen items are defective. Four items are chosen at random for inspection.
(i) Find the probability that none of the chosen items is defective.
(ii) Find the probability that the first defective item is found on the 4^{th} inspection.

$$\binom{n}{r} p^r q^{n-r}$$

F → finite no. of trials
I → independent
T → two outcomes
S → same probability

(i) $\binom{4}{0}(0.2)^0(0.8)^4$

$\frac{256}{625}$

Prob. of defective: 0.2
Prob not defective: 0.8

(ii) Consider:
None defective in first 3 inspections
and
defective item on 4th inspection

$$\binom{3}{0}(0.2)^0(0.8)^3 \times 0.2 = \frac{64}{625}$$

Two events E and F are independent.
If $P(\mathrm{E}) = \frac{1}{5}$ and $P(\mathrm{F}) = \frac{1}{7}$, find
(i) $P(\mathrm{E} \cap \mathrm{F})$ (ii) $P(\mathrm{E} \cup \mathrm{F})$.

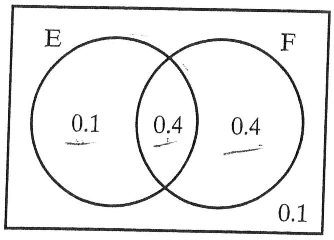
$* \; P(E \cup F) = P(E) + P(F) - P(E \cap F) \; *$

$*$ Indepent events $P(E \cap F) = P(E) \times P(F) \; *$

(i) $P(E \cap F) = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35}$

(ii) $P(E \cup F) = \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \frac{11}{35}$.

From the given Venn diagram, write down
(i) $P(\mathrm{E})$
(ii) $P(\mathrm{F})$
(iii) $P(\mathrm{E} \cup \mathrm{F})$
Now show that E and F are independent.
Find also $P(\mathrm{E}|\mathrm{F})$.



(i) $0.1 + 0.4 = 0.5$

(ii) $0.4 + 0.4 = 0.8$

(iii) $0.1 + 0.4 + 0.4 = 0.9$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{0.4}{0.8} = 0.5$

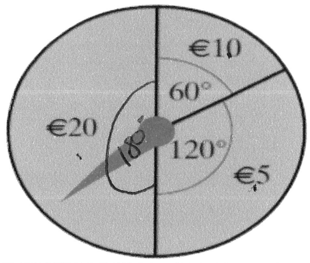
$P(E \cap F) = P(E) \times P(F)$

$0.4 = 0.5 \times 0.8$

$0.4 = 0.4$ ✓

Yes they are independent.

In the given wheel, you win the amount in the sector in which the arrow stops.
It costs €10 to play the game.
How much would you expect to win or lose if you play this game?
Explain why the game is not fair.



Outcome / Payout	Probability	Payout × probability
€10	$\frac{60}{360} = \frac{1}{6}$	$€10 \times \frac{1}{6} = \frac{10}{6}$ +
€5	$\frac{120}{360} = \frac{1}{3}$	$€5 \times \frac{1}{3} = \frac{5}{3}$ +
€20	$\frac{180}{360} = \frac{1}{2}$	$€20 \times \frac{1}{2} = 10$

Total: €$13\frac{1}{3}$

€13.33 − €10 = €3.33

Game is not fair, as if it was fair expected payout would be €0.
Here you are expected to win €3.33.