Math 6345 – AODEs

We now reconsider the 3 ODE:

(i) $\dot{x} = \mu - x^2$ saddle-node

(ii) $\dot{x} = \mu x - x^2$ transcritical $\Big\}$ bifurcations

(iii) $\dot{x} = \mu x - x^3$ pitchfork

but we now append the ODE $\dot{y} = -y$

and examine the nature of the CP's.

(i) $\dot{x} = \mu - x^2, \ \dot{y} = -y$

Crit pt will depend on μ

if $\mu \leq 0$ 1 CP or 0 CP if $\mu < 0$

$\mu > 0$ 2 CP at $(\pm\sqrt{\mu}, 0)$

$\mu = 0$ is the bifurcation pt.

for 1 CP when $\mu=0$ linear system won't work but we can integrate

$$\dot{x} = -x^2, \quad \dot{y} = -y$$

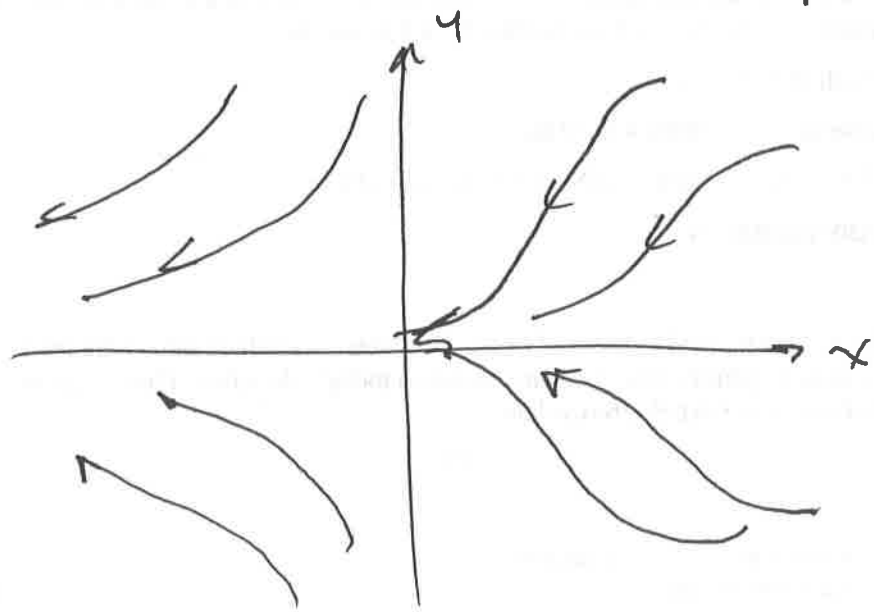
$$\frac{dy}{dx} = \frac{-y}{x^2}$$

integrate separately

$$x = \frac{1}{t+c_1}, \quad y = c_2 e^{-t}$$

or $\quad y = c\, e^{-1/x}$

Phase Portrait $\quad \mu = 0$



if $\mu > 0$ say $\mu = 1$

then crit pts are $(-1, 0)$ $(1, 0)$

for $\dot{x} = 1 - x^2, \; \dot{y} = -y$

$$D_x f = \begin{pmatrix} -2x & 0 \\ 0 & -1 \end{pmatrix}$$

<u>at $(-1,0)$</u> $D_x f = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ Eigenvalue $\lambda = -2, 1$

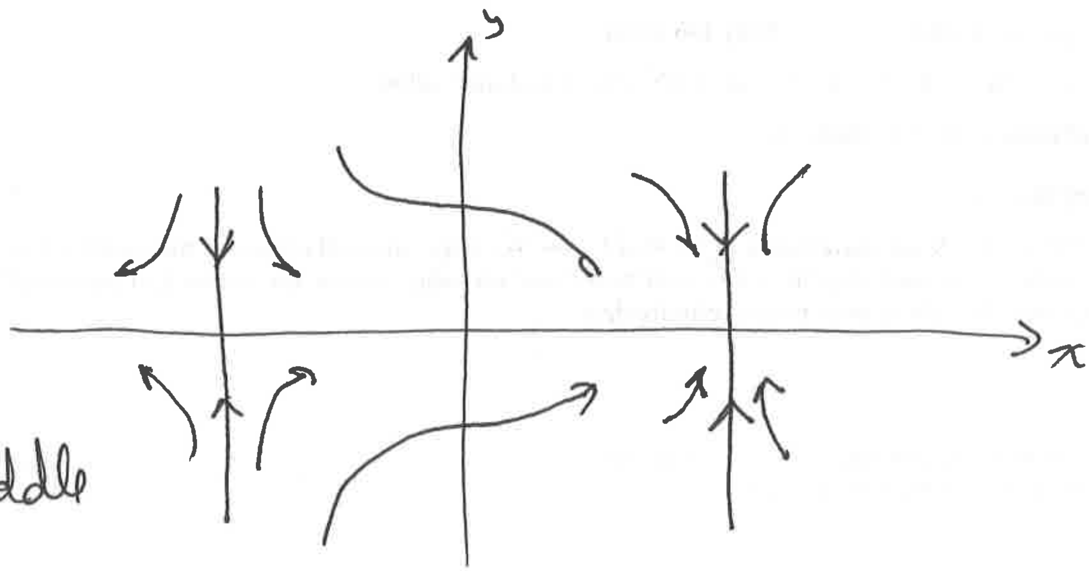
Saddle

sol^n $\bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$

<u>at $(1,0)$</u> $D_x f = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ Eigenvalues $\lambda = -2, -1$

stable node

sol^n $\bar{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$



Saddle

(ii) $\dot{x} = \mu x - x^2, \quad \dot{y} = -y$

In this case there are 2 crit pts.

$$(0,0) \;\&\; (\mu, 0)$$

$\mu = 0$ is still the bifurcation pt.

<u>Linearized Sys.</u> $\quad D_x f = \begin{pmatrix} \mu - 2x & 0 \\ 0 & -1 \end{pmatrix}$

at $(0,0)$ $\quad D_x f = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ Eigenvalues $\lambda = \mu, -1$

at $(\mu, 0)$ $\quad D_x f = \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix}$ $\quad$ Eigenvalues $\lambda = -\mu, -1$

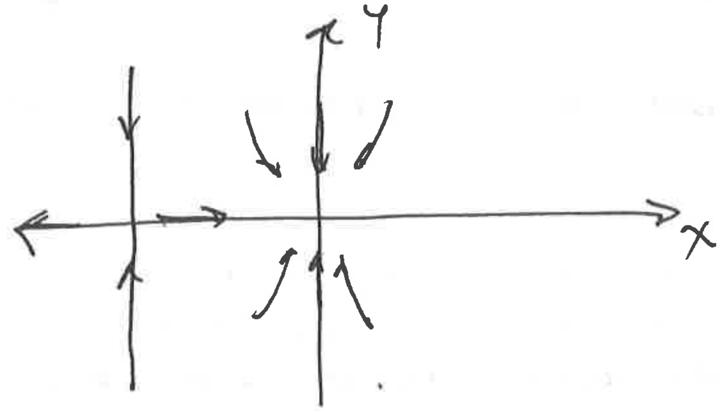
if $\mu < 0$ $\quad$ CP$(0,0)$ is a stable node

CP$(\mu, 0)$ is a saddle

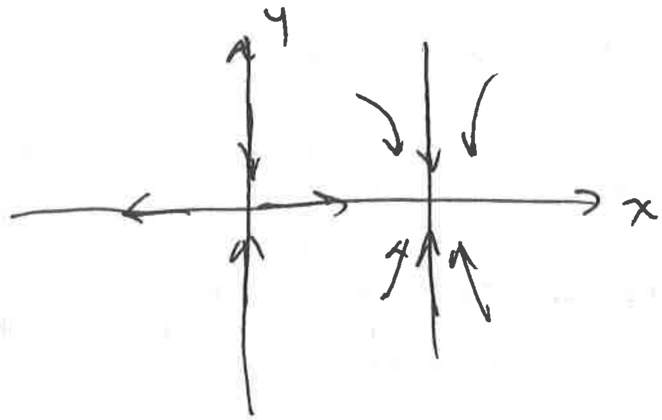
if $\mu > 0$ $\quad$ CP$(0,0)$ is a saddle

CP$(\mu, 0)$ is a stable node

$\mu < 0$



$\mu > 0$



(ii) $\dot{x} = \mu x - x^3, \quad \dot{y} = -y$

The crit pt $(0,0)$ persists
the other crit pt depends on μ

$\mu \leq 0$ CP $(0,0)$

$\mu > 0$ CPs: $(0,0)$ $(\pm\sqrt{\mu}, 0)$

$$D_x f = \begin{pmatrix} \mu - 3x^2 & 0 \\ 0 & -1 \end{pmatrix}$$

if $\mu < 0$ only CP is $(0, 0)$

so $D_x f = \begin{pmatrix} \mu & 0 \\ 0 & -1 \end{pmatrix}$ Eigenvalues $\lambda = \mu, -1$

$\mu < 0$ so stable node

if $\mu > 0$ then Eigenvalues $\left.\begin{matrix}\text{1 positive} \\ \text{1 negative}\end{matrix}\right\}$ saddle

& at $(\pm\sqrt{\mu}, 0)$

$D_x f = \begin{pmatrix} -2\mu & 0 \\ 0 & -1 \end{pmatrix}$ Eigenvalues $\lambda = -2\mu, -1$

both positive so stable node

$\mu < 0$ $\qquad\qquad$ $\mu > 0$