Fourier Series

Consider the series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

where L is a positive number and a_0, a_n and b_n constant coefficients. The question is: "How do we choose the coefficients as to give an accurate representation of $f(x)$?" Well, we use the following properties of $\cos\frac{n\pi x}{L}$ and $\sin\frac{n\pi x}{L}$

$$\int_{-L}^{L}\cos\frac{n\pi x}{L}dx = 0, \qquad \int_{-L}^{L}\sin\frac{n\pi x}{L}dx = 0, \tag{2}$$

$$\int_{-L}^{L}\cos\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{3}$$

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\ \sin\frac{m\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{4}$$

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\ \cos\frac{m\pi x}{L}dx = 0. \tag{5}$$

First, if we integrate (1) from $-L$ to L, then by the properties in (2), we are left with

$$\int_{-L}^{L} f(x)dx = \frac{1}{2}\int_{-L}^{L} a_0 dx = La_0,$$

from which we deduce

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx.$$

Next we multiply the series (1) by $\cos\frac{m\pi x}{L}$ giving

$$f(x)\cos\frac{m\pi x}{L} = \frac{1}{2}a_0\cos\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right).$$

Again, integrate from $-L$ to L. From (2), the integration of $a_0\cos\frac{m\pi x}{L}$ is zero, from (3), the integration of $\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero except when $n = m$ and further from (5) the integrations of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and n. This leaves

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = a_n\int_{-L}^{L}\cos^2\frac{n\pi x}{L}dx = La_n,$$

or

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx.$$

Similarly, if we multiply the series (1) by $\sin\frac{m\pi x}{L}$ then we obtain

$$f(x)\sin\frac{m\pi x}{L} = \frac{1}{2}a_0\sin\frac{m\pi x}{L} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right),$$

which we integrate from $-L$ to L. From (2), the integration of $a_0\sin\frac{m\pi x}{L}$ is zero, from (5) the integration of $\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is zero for all m and n and further from (4) the integration of $\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$ is zero except when $n = m$. This leaves

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = b_n\int_{-L}^{L}\sin^2\frac{n\pi x}{L}dx = Lb_n,$$

or

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

Therefore, the Fourier series representation of a function $f(x)$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

where the coefficients a_n and b_n are chosen such that

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx.$$

for $n = 0, 1, 2, \ldots$.