9 April 2011

The Rotating Mass Matrix: Some Fantasies

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1. Introduction

For the last decade, Hong-Mo Chan and colleagues have been arguing for a description of quark and lepton masses and mixings based upon something called the rotating mass matrix. It was motivated by models, but has evolved into a phenomenology largely detached from specific models. There is a review posted very recently on the arXiv (1103.5615), and this note will base itself on the contents of that review, hereafter called HMC.

The purpose here is to look at their idea in terms of a phenomenology less precise than their standards, but endowed with simplifying features. The goal is to dumb things down as much as possible, with an eye to spotting regularities that might lead back to models that possess such simplifying features. What follows will for the most part follow closely HMC. But in dealing with first-generation masses and mixings, which are incompletely described by HMC, we will be more adventurous and make some relatively wild guesses.

II. Fundamentals

We assume, as in HMC, that for each species of quark or lepton of given charge (e.g. top, charm, up comprise the "up-species") there exists a "mass matrix" $m_{ij} = M\,\alpha_i\,\alpha_j$. It, by definition, may be considered to be that part of the inverse Green's function of the fermion species,

$$G^{-1}(p)_{ij} = Z_i^{-1}(p^2)\left[\delta_{ij}\,\not p - M\,\alpha_i(p^2)\,\alpha_j(p^2)\right] \qquad (\mu^2 \equiv p^2)$$

which mixes left handed degrees of freedom with right handed degrees of freedom. The vector $\vec{\alpha}$, which lives in the 3d generation-space, is a function of scale μ. It will turn out that there is a special mass scale characterized by m = 7 MeV in what follows. For $\mu \gg$ m, as well as for $\mu \ll$ m, the vector is assumed to have the form

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \qquad 0 < |\alpha_1| \ll \alpha_2 \ll \alpha_3 \approx 1$$

The components α_i are assumed to satisfy renormalization group equations which relate their first derivatives with respect to log μ to functions which depend only on $\vec{\alpha}$. With the benefit of much hindsight, we will assume a very simple form for these evolution equations. We define the running variable t as

$$t = \ln \frac{\mu}{m}$$

The solution of the equations is assumed to be, for |t| >> 1, as follows

$$\alpha_1 \cong z\, e^{-|t|} \qquad \alpha_2 \cong e^{-\frac{1}{2}|t|} \qquad \alpha_3 \cong 1 \qquad z = e^{i\varphi}$$

The first component of alpha is allowed to be complex. But since the evolution equations for $\vec{\alpha}$, trivial to write down, are linear, this phase factor z does not evolve with scale. Hindsight reveals that introduction of a phase factor for α_2 leads, at the level of accuracy we consider, to no additional physical consequences, and is in fact a nuisance.

The rules for determining the three masses of each species, as well as their unit eigenvectors $\hat{u}_i$ in generation space, are as follows (taking the up-species as an example):

a) For $\mu = m_t$ the eigenvector $\hat{u}_t$ is just $\vec{\alpha}$ itself. The parameter $M = m_t$ is evidently the third-generation mass eigenvalue.

b) For $\mu = m_c$, first construct the piece of $\vec{\alpha}$ which is orthogonal to $\hat{u}_t$. Normalize it to unit value. This is defined as $\hat{u}_c$, the direction in generation space associated with the second-generation particle. The equation

$$m_c = M \left| \vec{\alpha}(m_c)^* \cdot \hat{u}_c \right|^2 = m_t \left| \vec{\alpha}^* \cdot \hat{u}_c \right|^2$$

is required to hold.

c) The first-generation unit eigenvector $\hat{u}_u$ Is defined as the cross product ($\hat{u}_c \times \hat{u}_t$). The equation

$$m_u = m_t \left| \vec{\alpha}(m_u)^* \cdot \hat{u}_c \right|^2$$

is required to hold.

The vector $\vec{\alpha}$ is found empirically by HMC to not depend upon species. Except for the phase factor z, we accept this feature here uncritically from the beginning. Then this procedure, when applied to the up and down quark species, allows the construction of the CKM matrix, which is the array of dot products of u_i^*(up) with u_j(down).

In the next section, we will construct the masses and the CKM mixings within this scenario for those quantities which are insensitive to CP violating phases. The remaining issues will be addressed in Section IV.

III. Results for the Second Generation Parameters

We first estimate the magnitudes of the CKM mixing-matrix elements. To do this, we must set down expressions for the mass eigenvectors. Omitting some small terms, we have for the up-species:

$$\begin{aligned} \hat{u}_t &= (0,\ 0,\ 1) & \alpha(m_t) &= (0,\ 0,\ 1) \\ \hat{u}_c &= (c_1/c_2,\ 1,\ 0) & \alpha(m_c) &\equiv (c_1,\ c_2,\ 1) \\ \hat{u}_u &= (1,\ -c_1^*/c_2,\ 0) & \alpha(m_u) &\equiv (u_1,\ u_2,\ 1) \end{aligned}$$

Similar expressions evidently hold for the down-species, although here we must, to first order, include the rotation of $\vec{\alpha}$ between the top-quark mass scale and the bottom-quark mass scale.

$$\hat{u}_b = (0,\ b_2,\ 1) \qquad \alpha(m_b) \cong (0,\ b_2,\ 1)$$

$$\hat{u}_s = (s_1/s_2,\ 1,\ -b_2) \qquad \alpha(m_s) = (s_1,\ s_2,\ 1)$$

$$\hat{u}_d = (1,\ -s_1^*/s_2,\ b_2\frac{s_1^*}{s_2}) \qquad \alpha(m_d) = (d_1,\ d_2,\ 1)$$

With these expressions for the basis vectors, we can go ahead and estimate the CKM elements. The diagonal elements to this level of accuracy are taken to be unity. The off-diagonal matrix elements are then found to be (again with some approximations):

$$V_{CKM} \cong \begin{pmatrix} 1 & \left(\frac{s_1}{s_2} - \frac{c_1}{c_2}\right) & -b_2\frac{c_1}{c_2} \\ \left(\frac{c_1^*}{c_2} - \frac{s_1^*}{s_2}\right) & 1 & b_2 \\ b_2\frac{s_1^*}{s_2} & -b_2 & 1 \end{pmatrix}$$

Note that the "unitarity triangle" relation

$$V_{td}^* + V_{ub} \cong V_{cb} V_{us}$$

is satisfied at this level of accuracy.

These numbers, to be summarized in Section VI, agree quite well with the measured magnitudes of the CKM elements.

We conclude this section with expressions for the second-generation masses. The charm mass is simply

$$m_c = M\,|\vec{\alpha}(m_c)^* \cdot \hat{u}_c|^2 \cong m_t c_2^2 = m_t\left(\frac{m}{m_c}\right)$$

The generalization is straightforward, leading to the results

$$m_c = \sqrt{m\,m_t} \qquad m_s = \sqrt{m\,m_b} \qquad m_\mu = \sqrt{m\,m_\tau}$$

Again, as documented in Section VI, these numbers are in satisfactory agreement with experiment.

IV. CP Violation Effects

The expression above for the CKM matrix shows that the phase of V_{ub} is simply the phase of the up-species $\alpha_1(\mu)$, evaluated for $\mu \gg M$. This, as we noted, can be regarded as a boundary condition for the vector $\vec{\alpha}$ in the ultraviolet region, $t \gg 1$. In order to account for CP violation, we must assume that this boundary condition is applied differently to the up-species and the down-species. We therefore assign different phases z_u and z_d as follows:

$$V_{ub} \hat{=} \frac{c_1}{c_2} b_2 \hat{=} \left| \frac{c_1}{c_2} b_2 \right| z_u \equiv c_2 b_2 z_u = c_2 b_2 e^{i\varphi_u}$$

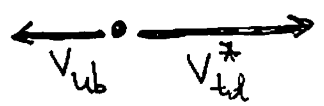
$$V_{td}^* \hat{=} s_1/s_2\, b_2 \hat{=} \left| s_1/s_2\, b_2 \right| z_d \equiv s_2 b_2 z_d = s_2 b_2 e^{i\varphi_d}$$

The effect of these phases is most easily visualized by turning the usual unitarity triangle upside down, resting it on the vertex γ. Experimentally, the angle associated with this vertex is almost exactly 90 degrees. Therefore, as shown in the figure below, the phases of z_u and z_d must satisfy the relation

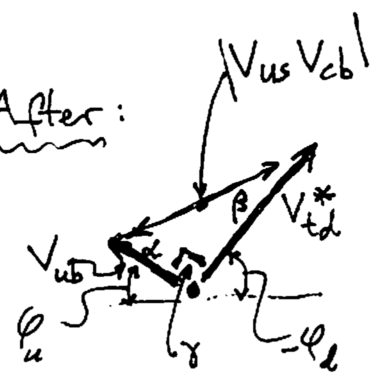
$$\varphi_u - \varphi_d = \frac{\pi}{2}$$

The evolution of the unitarity triangle is illustrated below:

Before ($\varphi_u = \varphi_d = 0$):



After:



HMC argue that the existence of these complex phases can be traced back to strong CP violation, namely a nonvanishing θ_{QCD}. We sympathize with this point of view, albeit without a detailed scenario of how θ_{QCD} finds its way into the boundary condition constraint on the phase of α_1 in the ultraviolet limit. Nevertheless there is a hint that γ being a right angle may be somehow related to a condition of maximal strong CP violation such as $\theta_{QCD} = \pi$.

V. First Generation Masses

The up-quark mass is given by the formula

$$m_u = m_t \left| \vec{\alpha}(m_u)^* \cdot \hat{u}_u \right|^2 \approx m_t \left| u_1^* - u_2 \frac{c_1^*}{c_2} \right|^2$$

Because the ratio m_u/m_t is so small, we must have

$$\frac{u_1^*}{u_2} = \frac{c_1^*}{c_2}$$

Were we to ignore phase factors, this requirement would lead to

$$\left|\frac{u_1}{u_2}\right| = u_2 = \left|\frac{c_1}{c_2}\right| = c_2 \qquad \Rightarrow \sqrt{\frac{m_u}{m}} = \sqrt{\frac{m}{m_c}}$$

Putting in the numbers gives much too small a result.

$$m_u \cong \frac{m^2}{m_c} \cong 50\ \mathrm{keV}$$

This is also true for the down-quark mass value

$$m_d \cong \frac{m^2}{m_s} \cong 300\ \mathrm{keV}$$

On the other hand, for the charged-lepton species, we have

$$m_e \cong \frac{m^2}{m_\mu} \cong 450\ \mathrm{keV}$$

This is a very good result, especially because there is no strong-CP-violation motivation for appending a phase factor onto the leptonic $\alpha_1(\mu)$, at any scale.

Therefore, we anticipate that the presence of the phases φ_u and φ_d will influence the values of m_u and m_d. Again, with hindsight, this can most easily and successfully be implemented by the simple prescription that in the infrared limit, $\mu \ll m$, α_1 is real for the up-species and for the down-species.

Consequently the expression for the up-quark mass becomes

$$m_u = m_t \left| |u_1| - u_2 \left|\frac{c_1}{c_2}\right| z_u \right|^2 \cong m_t u_2^2 \left| u_2 - c_2 z_u \right|^2$$

Here we have used the fact that

$$|u_1| = u_2^2 \qquad |c_1| = c_2^2$$

But the constraint

$$u_2 = c_2$$

still must hold, even in the presence of the phase φ_u. Therefore

$$\frac{m_u}{|1 - z_u|^2} \cong m_t c_2^4 \cong m_t \left(\frac{m}{m_c}\right)^2 = \frac{m^2 m_t}{m m_t} = m$$

We conclude that

$$m_u = m|1 - e^{i\varphi_u}|^2$$
$$m_d = m|1 - e^{i\varphi_d}|^2$$

The ratio m_u / m_d is not determined. But the sum of m_u and m_d is rather well determined. To see this, write

$$\varphi_u = \frac{\pi}{4} - \delta$$
$$\varphi_d = -\frac{\pi}{4} - \delta$$

For the typical estimate

$$\frac{m_u}{m_d} \sim \frac{1}{2}$$

we obtain, after some algebra,

$$m_u + m_d \approx m(4 - 2\sqrt{2}) \approx 1.2m \approx 8\,\text{MeV}$$

$$\delta \approx 0{,}15$$

This is in quite good agreement with experiment.

VI. Numbers

We here summarize the numerical consequences of the above exercise, with a slight amount of care toward accuracy. A much more professional job could be done, but that is beyond the scope of this note.

The input parameters are the third-generation masses, taken from the PDG compilation. The mysterious mass scale m is here chosen, with some hindsight, as a reasonable estimate, but not as a best fit:

m_t = 171 GeV

m_b = 4.2 GeV

m_τ = 1.78 GeV

m = 6.8 MeV

In addition, the unitarity-triangle angle γ, experimentally determined to be 90 degrees within errors of a few degrees, is chosen here as an input parameter.

The output second generation masses are given by the geometric mean of the third generation masses and m :

m_c = 1.08 GeV (1.3 $\pm$ 0.1 GeV)

m_s = 170 MeV (105 $\pm$ 30 MeV)

m_μ = 110 MeV (106 MeV)

(The quantities in parentheses are the PDG book values.)

The electron mass is determined by assuming that m is the geometric mean of it and the muon mass:

m_e = 0.44 MeV (0.51 MeV)

After strong CP violating effects are considered, the sum of the up-quark and the down-quark (current-algebra) masses is given by

$m_u + m_d$ = 1.2 m = 8.2 MeV (7.5 $\pm$ 2.5 MeV)

The ratio of the masses is not determined.

The magnitudes of the off-diagonal CKM matrix elements are determined as follows:

$|V_{cb}| \cong -|V_{ts}|$ = .040 (.041)

$|V_{ub}|$ = .0032 (.0039)

$|V_{td}|$ = .0080 (.0081)

The formulae determining these values are as follows:

$$|V_{cb}|^2 = \frac{m}{m_t} \qquad |V_{ub}|^2 = \frac{m^2}{m_b m_c} \qquad |V_{td}|^2 = \frac{m^2}{m_b m_s}$$

Once one of the angles of the unitarity triangle is specified, the value of $|V_{us}|$ will be determined. We choose the angle γ, and take it to be 90 degrees. Upon using the unitarity triangle relation, this gives an intriguingly simple rule for determining V_{us}:

$$|V_{cd}|^2 \cong |V_{us}|^2 \cong \left(\frac{m}{m_c} + \frac{m}{m_s}\right)$$

This angle γ appears to be directly related to half the value of θ_{QCD}, although there are in my opinion still many missing details to check out. If there are no difficulties, this would imply that the final output of these considerations is

$$\theta_{QCD} \overset{?}{=} \pi$$

VII. Comments

The above numbers rest on very shaky foundations. Fortunately, there are many things remaining to be considered, which can test this fragile structure.

1. The neutrino masses and mixings have not been addressed. The MNS matrix governing neutrino mixing is well represented by the tribimaximal form. This in turn can be viewed as the rotation taking the α-triad containing the massive eigenvector (1, 0, 0) to the α-triad containing the massive eigenvector $(1, 1, 1)/\sqrt{3}$. In the early HMC papers this is just the total rotation angle between the α-triad evaluated in the far ultraviolet, and the α-triad evaluated in the far infrared. In these notes this net rotation does not seem to occur. But it is possible that the orientation of the triad at the mysterious scale m = 7 MeV is tribimaximal relative to the extreme ultraviolet and infrared limits.
2. The dynamics behind the rotation of $\vec{\alpha}$ needs to be explicated. In my opinion, one not necessarily shared by HMC, this is most cogently implemented by assuming, in addition to the standard-model generic electroweak Higgs sector, the existence of a set of low mass pseudo-Goldstone bosons I will call familons. These familons probably should be modeled in a style as similar as possible to what appears in the HMC papers. What seems to me to be a minimal scheme is to assume that there is a self-conjugate nonet or octet of such spinless, neutral, colorless, electroweak-singlet states. Most, but not necessarily all, of these should have low but not vanishing masses, which may well be hierarchical. In particular, based on what I see in the models considered by HMC, all six off-diagonal states may be expected to be familons. Some or all of the three, or perhaps two, remaining "diagonal" nonet states might have very large masses, larger than or comparable to the top quark mass, although it is probably desirable to have at least one of these diagonal states play a role similar to that of the Peccei-Quinn axion. It is also a natural option to allow this familon sector to interact strongly with itself. However it should only be allowed to communicate with the electroweak sector via a quartic coupling, quadratic in the familon fields and quadratic in the electroweak Higgs fields. And, as will be mentioned again below, the Yukawa couplings of familons to quarks and leptons need to be suppressed for phenomenological reasons. This can be implemented in good part by assuming a G-parity-like symmetry. For all practical purposes, the action should remain invariant under replacement of all familon fields by their negatives, with the fermion fields staying unchanged.
3. One possible consequence of such familon states is in dark-matter phenomenology. They could link not only to axions as dark matter, but also to "hidden-sector" models containing low-mass standard-model singlet states.
4. The heaviest of the familon states might have a mass related to the mysterious mass scale m = 7 MeV. It could decay promptly and invisibly into lighter familon states, making experimental detection very challenging. But there might be final states containing photons (or electron-positron pairs?); this option deserves detailed consideration.
5. There probably are flavor off-diagonal couplings of familon-antifamilon pairs to fermion-antifermion pairs, with current-current interactions perhaps constrained via Adler-Weisberger-

like current-algebra low energy theorems. These may be the most important processes which phenomenologically limit (or rule out) this set of ideas. Typical processes that come to mind are

$$\tau \to \mu + a + \bar{a} \qquad b \to s + a + \bar{a}$$
$$\mu \to e + a + \bar{a} \qquad K \to \pi + a + \bar{a} \qquad Z \to a + \bar{a}$$

6. Of course precision tests like g – 2, which involve these familon-fermion couplings in virtual processes, will be very constraining.
7. On the optimistic side, the anomalous B_s mixing reported by the D0 collaboration might be mediated by familon-antifamilon intermediate states. This deserves careful analysis.
8. Likewise, the anomalous top-antitop forward-backward asymmetry reported by CDF might be interpretable as a rather strong mixing with a strongly interacting familon-antifamilon sector. An analysis based on coupled-channel S-matrix theory might possibly lead to interference effects between partial waves which create the asymmetry.
9. Finally it should be noted that the aforementioned quartic coupling which connects the familon sector to the standard-model Higgs sector appears to allow the standard-model Higgs to decay invisibly into familon-antifamilon pairs, with quite possibly a width large compared to the total standard-model width. If this occurs, all the visible decay modes of the Higgs which are on the LHC menu could have their branching ratios diluted by a factor ten or even one hundred. There might even be a nightmare case where the Higgs becomes the "sigma meson of the weak interaction" and has a width comparable to its mass. If anything like this were to become real, the rationale for constructing a next linear collider would be greatly strengthened.

I have only looked at this familon hypothesis at the hand-waving level. To be sure it is phenomenologically very dangerous, and the rotating mass matrix scheme itself remains very fragile. But before dismissing all this as wrong, I want to know either (1) why it is obviously theoretically inconsistent, or (2) which experiment or experiments rule it out.