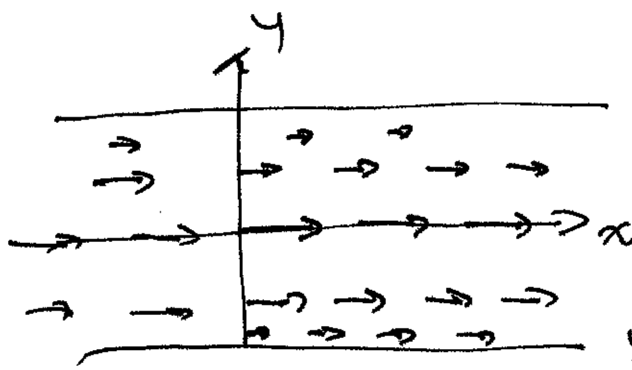
Math 2371 – Calc 3

Vector Fields

$\vec{F} = \langle f(x,y), g(x,y) \rangle$ 2-D

$\vec{F} = \langle f(x,y,z), g(x,y,z), h(x,y,z) \rangle$ 3-D

Ex $\bar{F} = \langle 1-y^2, 0 \rangle$ for $-1 \le y \le 1$



$y=1$ $\bar{F}(x,1) = \langle 0, 0 \rangle$

$\bar{F}(x,0) = \langle 1, 0 \rangle$

$y=-1$ $\bar{F}(x,-1) = \langle 0, 0 \rangle$

Kinda like the flow in a tube

Gradient Vector Fields

$\bar{F} = \bar{\nabla} f$ f – potential function so if $f = xy^2 + x - 3y$

$\bar{\nabla} f = \langle y^2+1, 2xy - 3 \rangle = \bar{F}$

so Given $\bar{F}$ ~~how~~ do we know if

it's a gradient VF

Ex: is $\bar{F} = \langle -y, x \rangle$ one?

or $\bar{F} = \langle x, y \rangle$?

if so $\bar{\nabla} f = \langle -y, x \rangle$

a $f_x = -y$, $f_y = x$

cross der $f_{xy} = -1$ $f_{yx} = 1 \neq -1$ so this one no

$f_x = x$ $f_y = y$

$f_{xy} = f_{yx} = 1$ so yes!

then we integrate

$f_x = x \Rightarrow \quad f = \frac{x^2}{2} + A(y)$

$f_y = y \Rightarrow \quad f = \frac{y^2}{2} + B(x)$

Are these ever pieces the same? no!

Now choose $A(y) = y^2/2 \quad B(x) = \frac{x^2}{2}$

so $\quad f = \frac{1}{2}x^2 + \frac{1}{2}y^2 + c$

Ex $\quad \bar{F} = \langle 3x^2y + y^3 + 1,\ x^3 + 3xy^2 \rangle$

if gradient $\quad f_x = 3x^2y + y^3 + 1,\ f_y = x^3 + 3xy^2$

$f_{xy} = \cancel{6xy}$ $\quad f_{yx} = 3x^2 + 3y^2$

$3x^2 + 3y^2$ $\quad$ so yes

~~us~~ view $\int$ $\quad f = x^3y + xy^3 + x + A(y)$

$f = x^3y + xy^3 + B(x)$

so same piece is $x^3y + xy^3$

choose $B(x) = x$ $A(y) = 0$

$f = x^3y + xy^3 + x + c$

3-D

$\vec{F} = \langle yz^2 + y,\ xz^2 + x,\ 2xyz - 3 \rangle$

is this a gradient VF

if so $f_x = yz^2 + y$

$f_y = xz^2 + x$

$f_z = 2xyz - 3$

need to check

$f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \quad f_{yz} = f_{zy}$

we can do this all at once!

Gradient Operator

$$\bar{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Divergence of a Vector Field

if $\bar{F} = \langle f, g, h \rangle$

$$\bar{\nabla} \cdot \bar{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle$$

$$= f_x + g_y + h_z$$

a measure of flow

Curl of a Vector Field

if $\bar{F} = \langle f, g, h \rangle$

$$\bar{\nabla} \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

a measure of rotation

so if $\bar{F} = \langle yz^2+y,\ xz^2+x,\ 2xyz-3 \rangle$

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial}{\partial x}(yz^2+y) + \frac{\partial}{\partial y}(xz^2+x) + \frac{\partial}{\partial z}(2xyz-3)$$

$$= 0 + 0 + 2xy$$

$$\bar{\nabla}\times\bar{F} = \begin{vmatrix} \vec{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2+y & xz^2+x & 2xyz-3 \end{vmatrix}$$

$$= \langle 2xz - 2xz,\ -(2yz-2yz),\ z^2-z^2 \rangle$$

$$= \langle 0, 0, 0 \rangle$$

if $\bar{\nabla}\times\bar{F} = \bar{0}$ then $\bar{F}$ is a gradient VF.

we use the term "conservative" VF.

so f exists

$f_x = yz^2 + y$ $\qquad$ $f = xyz^2 + xy + A(y,z)$

$f_y = xz^2 + x$ $\qquad$ $f = xyz^2 + xy + B(x,z)$

$f_z = 2xyz - 3$ $\qquad$ $f = xyz^2 - 3z + C(x,y)$

choose $A = B = -3z \quad C = xy$

so $\quad f = xyz^2 + xy - 3z + c \quad$ (some const)