Chain Rule

Math 4315 - PDEs - Calc III review

Chain rule

Consider

$$u = x^2 y \tag{1}$$

and the change of variables

$$x = e^r, \qquad y = \sin r. \tag{2}$$

Directly substituting (2) into (1) gives

$$u = e^{2r} \sin r. \tag{3}$$

Since $u = u(r)$ it makes sense to take a derivative with respect to the single variable r and so

$$\frac{du}{dr} = 2e^{2r} \sin r + e^{2r} \cos r \tag{4}$$

In calculus III, we were introduced to the two types of chain rules.

Type 1 Chain Rule

If $u = u(x, y)$ and $x = f(r)$ and $y = g(r)$ then $u = u(r)$ and

$$\frac{du}{dr} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dr} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dr} \tag{5}$$

For the preceding example

$$\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial u}{\partial y} = x^2, \quad \frac{dx}{dr} = e^r, \quad \frac{dy}{dr} = \cos r, \tag{6}$$

and from (5)

$$\begin{aligned} \frac{du}{dr} &= 2xy \cdot e^r + x^2 \cdot \cos r \\ &= 2e^r \sin r e^r + e^{2r} \cos r \end{aligned} \tag{7}$$

giving exactly (4).

Consider the same u in (1) but now

$$x = e^r \cos s, \quad y = e^r \sin s \tag{8}$$

Directly substituting gives

$$u = e^{3r} \cos^2 s \sin s \tag{9}$$

and since $u = u(r, s)$ it makes sense to take derivatives with respect to r and s and here

$$u_r = 3e^{3r} \cos^2 s \sin s, \quad u_s = e^{3r} \left(-2 \cos s \sin^2 s + \cos^3 s\right) \tag{10}$$

Type 2 Chain Rule

If $u = u(x, y)$ and $x = f(r, s)$ and $y = g(r, s)$ then $u = u(r, s)$ and

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}, \\ \frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}. \end{aligned} \tag{11}$$

For the preceding example

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2xy, & \frac{\partial u}{\partial y} &= x^2, \\ \frac{\partial x}{\partial r} &= e^r \cos s, & \frac{\partial y}{\partial r} &= e^r \sin s, \\ \frac{\partial x}{\partial s} &= -e^r \sin s, & \frac{\partial y}{\partial s} &= e^r \cos s, \end{aligned} \tag{12}$$

and so from (21)

$$\begin{aligned} \frac{\partial u}{\partial r} &= 2xy \cdot e^r \cos s + x^2 \cdot e^r \sin s, \\ \frac{\partial u}{\partial s} &= -2xy \cdot e^r \sin s + x^2 \cdot e^r \cos s, \end{aligned} \tag{13}$$

and further substitution of x and y from (8) gives (10).

u Unknown

Suppose that in general $u(x, y)$ is unknown but suppose we knew that

$$u(x, y) = r^2 \tag{14}$$

if

$$x = r \quad y = -r, \tag{15}$$

namely

$$u(r,-r) = r^2 \tag{16}$$

could we differentiate (16)? Namely, can we

$$\frac{d}{dr}u(r,-r) = \frac{d}{dr}r^2 \tag{17}$$

Here we will use the chain rule (5) except since we don't know the form of u will left the derivatives alone. So

$$\begin{aligned}\frac{du}{dr} &= \frac{\partial u}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dr}\\ &= \frac{\partial u}{\partial x}\cdot 1 + \frac{\partial u}{\partial y}\cdot(-1) = 2r\end{aligned} \tag{18}$$

Consider $u = u(x,y)$ (now unkown) and the change of variables

$$r = x + y, \qquad s = x - y. \tag{19}$$

Could we calculate u_x and u_y in terms of u_r and u_s. Again we will use a type 2 chain rule an in this case

$$\begin{aligned}u_x &= u_r \cdot r_x + u_s \cdot s_x\\ u_y &= u_r \cdot r_y + u_s \cdot s_y\end{aligned} \tag{20}$$

From (19) r_x, r_y, s_x and s_y are all easy to calculate and so

$$u_x = u_r + u_s, \tag{21a}$$

$$u_y = u_r - u_s. \tag{21b}$$

If the change of variables is

$$r = 2xy, \qquad s = x^2 + y^2. \tag{22}$$

the from

$$\begin{aligned}u_x &= u_r \cdot r_x + u_s \cdot s_x\\ u_y &= u_r \cdot r_y + u_s \cdot s_y\end{aligned} \tag{23}$$

then u_x and u_y become

$$u_x = u_r \cdot 2y + u_s \cdot 2x, \tag{24a}$$

$$u_y = u_r \cdot 2x + u_s \cdot 2y. \tag{24b}$$

Suppose we wanted to have the right hand side of (24) to only involve r and s only, we would have to solve (22) for x and y explicitly. As it turns out we can giving

$$x = \frac{1}{2}\left(\sqrt{s+r} + \sqrt{s-r}\right), \quad y = \frac{1}{2}\left(\sqrt{s+r} - \sqrt{s-r}\right) \tag{25}$$

and these can be substituted into (24).