# Math 6345 AODE Lect. 1

Solve $\dfrac{d\bar{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \bar{x}$

So $\begin{vmatrix} \lambda-2 & 0 \\ -1 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-2) = 0$

so $\lambda = 1, 2$

if $\lambda = 1$ $\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow e_1 = 0$ $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow e_1 - e_2 = 0$ $\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

sol$^n$ $\bar{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

$= \begin{pmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$= \underline{\Phi}(t)\,\bar{c}$ when $\underline{\Phi}$ - fundemental matrix then we asked

if $\frac{dx}{dt} = ax \Rightarrow x = ce^{at}$

then $\frac{d\bar{x}}{dt} = A\bar{x} \Rightarrow \bar{x} = e^{At}\bar{c}$

so is $e^{At}\bar{c}_1 = \Phi(t)\bar{c}_2$ $\qquad \bar{c}_1 \neq \bar{c}_2$

Given an I.C. so $\bar{x}(0) = x_0$

then $e^{A(0)}\bar{c}_1 = \bar{x}_0$ & $\Phi(0)\bar{c}_2 = x_0$

must be the same

So we need to make sense of $e^{At}$

Def$^n$ We define the matrix exponential as

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

where $A$ is an $n \times n$ real valued matrix.

so $e^{A[0]} = e^{[0]} = I$

so $\bar{c}_1 = \bar{x}_0$ & $\Phi(0) \cdot \bar{c}_2 = x_0 = c_1$

$\Rightarrow \quad \bar{c}_1 = \bar{x}_0 \qquad \bar{c}_2 = \Phi^{-1}(0)\,\bar{x}_0$

& $\bar{x} = e^{At}\bar{x}_0$ & $\bar{x} = \Phi(t)\,\Phi^{-1}(0)\,\bar{x}_0$

& $\boxed{e^{At} = \Phi(t)\,\Phi^{-1}(0)}$

So let's consider the previous example

$$\Phi = \begin{pmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix} \qquad \Phi^{-1} = -\frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 0 \end{pmatrix}$$

a $\Phi^{-1} = \begin{pmatrix} -e^{-t} & e^{-t} \\ +e^{-2t} & 0 \end{pmatrix} \qquad \Phi^{-1}(0) = \begin{pmatrix} -1 & 1 \\ +1 & 0 \end{pmatrix}$

and $\Phi(t)\,\Phi^{-1}(0) = \begin{pmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ +1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} e^{2t} & 0 \\ e^{2t}-e^{t} & e^{t} \end{pmatrix}$$

to show its a sol$^{n}$

$$\frac{d}{dt}\,\Phi(t)\,\Phi^{-1}(0) = \begin{pmatrix} 2e^{2t} & 0 \\ 2e^{2t}-e^{t} & e^{t} \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ e^{2t}-e^{t} & e^{t} \end{pmatrix} =$$ yes

so could we ~~have~~ found $e^{At}$ directly from definition

$$e^{\left[\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right]t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}t + \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}\frac{t^2}{2!} + \begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix}\frac{t^3}{3!} + \dots$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}^3 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2t+\frac{4t^2}{2!}+\frac{8t^3}{3!} & 0 \\ t+\frac{3t^2}{2!}+\frac{7t^3}{3!} & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & 0 \\ ? & e^{t} \end{pmatrix}$$

And this was a pretty straight forward example

Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \bar{x} \qquad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t + \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} \frac{t^2}{2!} + \cdots$$

$$= \begin{pmatrix} 1+at+\frac{at^2}{2!}+\cdots & 0 \\ 0 & 1+dt+\frac{dt^2}{2!}+\cdots \end{pmatrix} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{dt} \end{pmatrix}$$

Ex $\quad \dfrac{dx}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$

$$\begin{vmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{vmatrix} = 0 \qquad \begin{aligned} &\lambda^2 - 4\lambda + 4 - 1 = 0 \\ &\lambda^2 - 4\lambda + 3 = 0 \\ &(\lambda-1)(\lambda-3) = 0 \qquad \lambda = 1, 3 \end{aligned}$$

lets go directly to $e^{At}$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^4 = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^5 = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} + \frac{41t^4}{4!} + \cdots & \underline{\qquad\qquad} \\ t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \frac{40t^4}{4!} + \cdots & \underline{\qquad\qquad} \end{pmatrix}$$

and what are these ?

However, note

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

so
$$e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}t} = e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}t}$$

$$= e^{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}t} \cdot e^{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}t}$$

$2^{nd}$ exponential with $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{pmatrix}$

$$e^{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}t} = \begin{pmatrix} 1 + t + \frac{2t^2}{2!} + \frac{4t^3}{3!} + \frac{8t^4}{4!} & t + \frac{2t^2}{2!} + \frac{4t^3}{3!} + \frac{8t^4}{4!} + \cdots \\ & \end{pmatrix}$$

$$t + \frac{2t^2}{2!} + \frac{4t^3}{3!} + \frac{8t^4}{4!} + \cdots \qquad \frac{1}{2}\left(2t + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \cdots\right)$$

$$= \frac{1}{2}e^{2t}$$

$$\frac{1}{2}\left(2 + 2t + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \cdots\right)$$

$$\frac{1}{2}\left(1 + e^{2t}\right)$$

However, consider

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t}$$

since $\begin{pmatrix}1&1\\0&0\end{pmatrix}^n = \begin{pmatrix}1&1\\0&0\end{pmatrix}$ from the def$^n$ of the matrix exp

$$e^{\begin{pmatrix}1&1\\0&0\end{pmatrix}t} = \begin{pmatrix}e^t & e^t-1\\0&1\end{pmatrix}$$

But $\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix}$

so $e^{\begin{pmatrix}1&0\\0&0\end{pmatrix}t} = \begin{pmatrix}e^t&0\\0&1\end{pmatrix}$ $\quad e^{\begin{pmatrix}0&1\\0&0\end{pmatrix}t} = \begin{pmatrix}1&t\\0&1\end{pmatrix}$

$$\because \begin{pmatrix}0&1\\0&0\end{pmatrix}^n = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

and

$$\begin{pmatrix}e^t & e^t-1\\0&1\end{pmatrix} \neq \begin{pmatrix}e^t&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&t\\0&1\end{pmatrix}$$