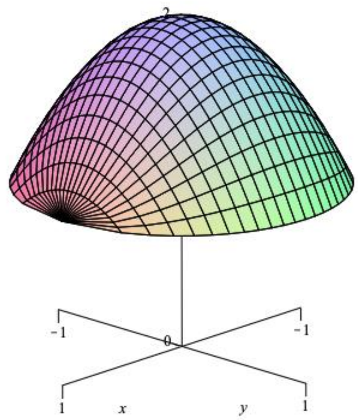
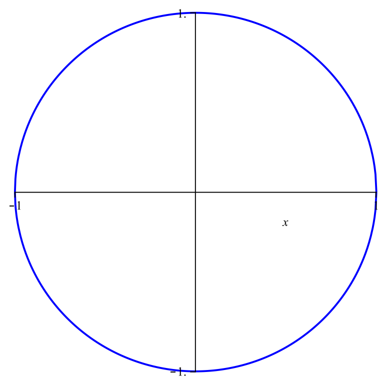
Calculus 3 - Double Integrals - Polar

Last class we considered the problem of finding the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$





As the region of integration is a circle, we indicated that maybe using polar coordinates might be the best way to tackle this problem. Recall we introduced polar coordinates where

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{1}$$

and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x}. \tag{2}$$

We said that sweeping out the region of the circle that

$$r = 0 \to 1, \quad \theta = 0 \to 2\pi \tag{3}$$

and asked, how would the double integral change if we used r and θ instead of x and y? Would it becomes easier?

Double Integrals in Polar Coordinates

We consider the double integral

$$V = \iint_R f(x,y)dA. \tag{4}$$

This integral has three main parts:

1. the integrand
2. dA
3. limits

1. The integrand

For this part, we simplify substitute in

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{5}$$

into $f(x,y)$ and simplify. So, in general,

$$V = \iint_R f(x,y)dA = \iint_R f(r\cos\theta, r\sin\theta)dA. \tag{6}$$

So if the integral was say

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dydx. \tag{7}$$

then

$$V = \iint_R \sqrt{1-r^2}\, dA \tag{8}$$

2. dA

In cartesian coordinates, this is $dA = dxdy$. What about in terms of dr and $d\theta$? Let us consider where the $dxdy$ came from. It came from a small area element

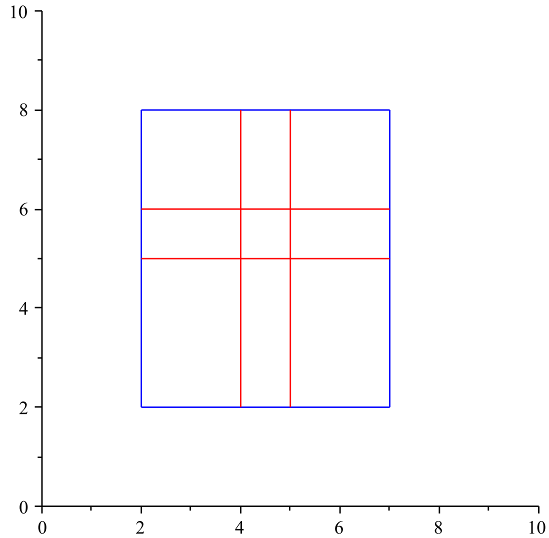


Figure 1: Cartesian region of integration

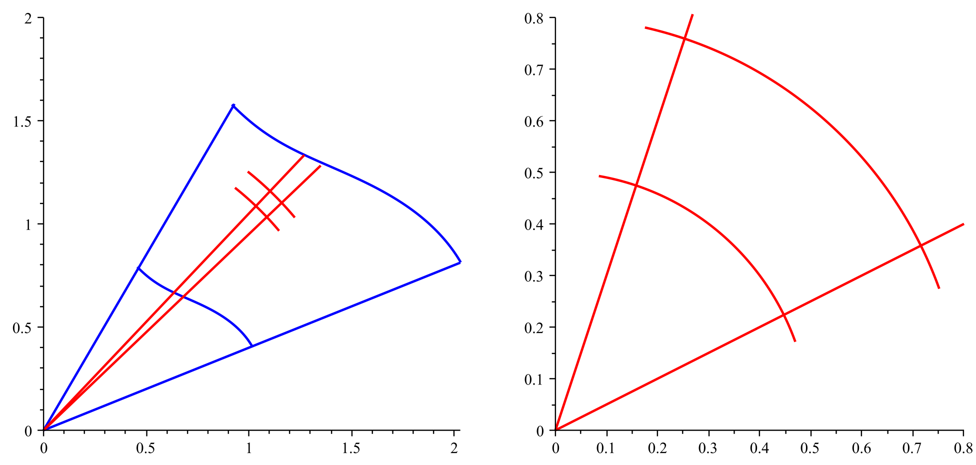


Figure 2: Polar region of integration

Now from our arc length formula where $s = r\theta$ then $ds = r\,d\theta$. The change is r is dr and we have

$$dA = rd\theta \times dr \tag{9}$$

3. Limits of Integration

These ultimately come from the picture of the region itself.

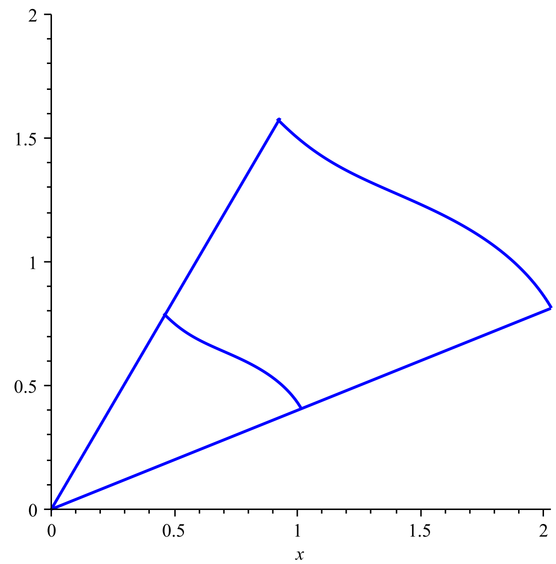


Figure 3: Cartesian region of integration

so

$$\int_{\alpha}^{\beta} \int_{r_i(\theta)}^{r_o(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta \tag{10}$$

where $r = r_i(\theta)$ is the inner curve and $r = r_o(\theta)$ is the outer curve.

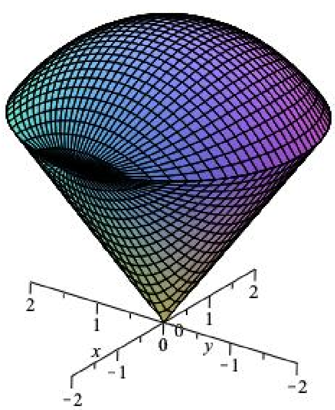
Example 1. Find the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

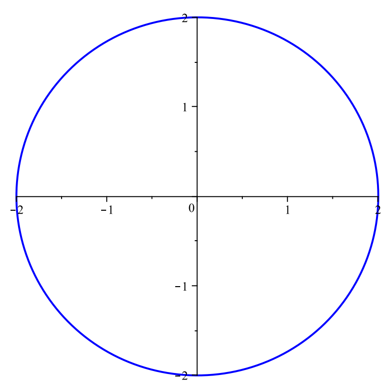
Soln. The surface is given by $z = 2 - r^2$ (this is the integrand). We are integrating over a circle of radius 1 so the volume we seek is given by

$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r - r^3 dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2} r^2 - \frac{1}{4} r^4 \Big|_0^1 d\theta \\
&= \frac{1}{4} \int_0^{2\pi} d\theta \\
&= \frac{1}{4} \theta \Big|_0^{2\pi} = \frac{\pi}{2}
\end{aligned} \tag{11}
$$

Example 2. Find the volume between the cone $z = \sqrt{x^2 + y^2}$ and the half sphere $z = \sqrt{8 - x^2 - y^2}$.

Soln. The surfaces are





The intersection of the two surface will give the region of integration so

$$
\sqrt{8 - x^2 - y^2} = \sqrt{x^2 + y^2} \quad \Rightarrow \quad 8 - x^2 - y^2 = x^2 + y^2 \tag{12}
$$

or

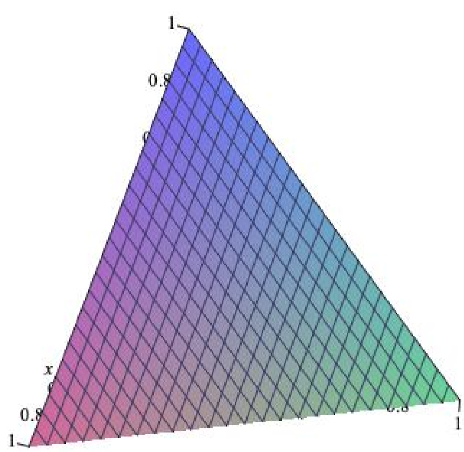
$$x^2 + y^2 = 4 \tag{13}$$

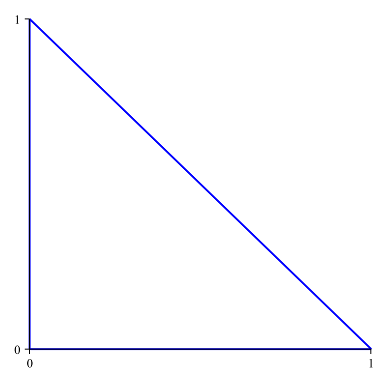
As there are two surfaces, there will be two in the integrand. The setup is as follows:

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^2 \left(\sqrt{8 - r^2} - r \right) r dr d\theta \\
&= \int_0^{2\pi} -\tfrac{1}{3} \left(8 - r^2\right)^{2/3} - \tfrac{1}{3} r^3 \Big|_0^2 d\theta \\
&= \tfrac{16}{3} (\sqrt{2} - 1) \int_0^{2\pi} d\theta \\
&= \frac{16}{3} (\sqrt{2} - 1) 2\pi
\end{aligned} \tag{14}$$

Example 3. Find the volume of the tetrahedran bound by the planes $x + y + z = 1$ $x = 0$, $y = 0$ and $z = 0$.

Soln. The surfaces are





The setup for this problem is

$$\int_0^{\pi/4} \int_0^{\frac{1}{\cos\theta+\sin\theta}} (1 - r\cos\theta - r\sin\theta)\, r dr d\theta \tag{15}$$

Clearly, Cartesian is the way to go.

Example 4. Pg. 995, #18 Evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} x dy dx \tag{16}$$

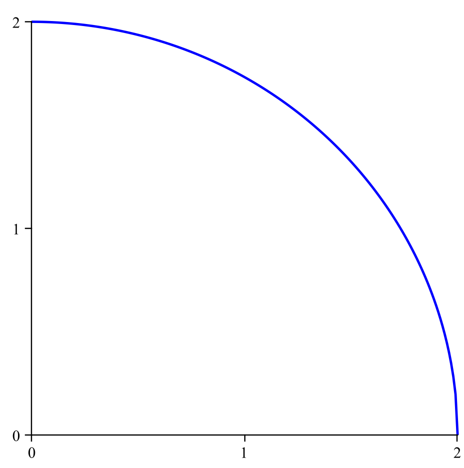
Soln.

We first draw the region. We integrate with respect to y first so we go from a bottom curve to a top curve and in this case

$$y = 0 \quad \Rightarrow \quad y = \sqrt{4 - x^2} \tag{17}$$

Next we integrate with respect to x and so this is a left point and right point

$$x = 0 \quad \Rightarrow \quad x = 2 \tag{18}$$



Now the setup

$$\begin{aligned}\int_0^{\pi/2}\int_0^2 r\cos\theta\; rdrd\theta &= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\; drd\theta \\ &= \int_0^{\pi/2}\int_0^2 \frac{1}{3}r^3\Big|_0^2 \cos\theta\; d\theta \\ &= \frac{8}{3}\sin\theta\Big|_0^{\pi/2} = \frac{8}{3}.\end{aligned} \tag{19}$$

Example 4. Pg. 995, #24 Evaluate

$$\int_0^4\int_0^{\sqrt{4y-y^2}} x^2 dxdy \tag{20}$$

Soln.

We first draw the region. We integrate with respect to x first so we go from a left curve to a right curve and in this case

$$x = 0 \quad \Rightarrow \quad x = \sqrt{4y-y^2}. \tag{21}$$

Next we integrate with respect to y and so this is a bottom point and top point

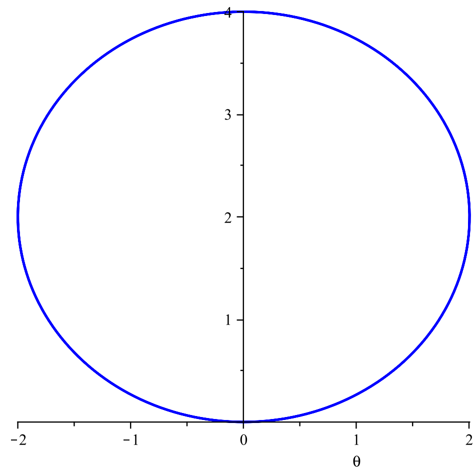
$$y = 0 \quad \Rightarrow \quad y = 4. \tag{22}$$

To get an idea of what the right curve is

$$x = \sqrt{4y-y^2} \quad \Rightarrow \quad x^2 + y^2 - 4y = 0, \tag{23}$$

so

$$r^2 - 4r\sin\theta = 0 \quad \Rightarrow \quad r = 4\sin\theta. \tag{24}$$



Now the setup

$$\int_0^{\pi/2}\int_0^{4\sin\theta}(r\cos\theta)^2\ rdrd\theta = \int_0^{\pi/2}\int_0^{4\sin\theta} r^3\cos^2\theta\ rdrd\theta$$
$$= \int_0^{\pi/2}\frac{1}{4}r^4\Big|_0^{4\sin\theta}\cos^2\theta\ d\theta$$
$$= 4^3\int_0^{\pi/2}\sin^4\theta\cos^2\theta\ d\theta$$
$$= 4^3\cdot\frac{\pi}{32} = 2\pi. \tag{25}$$