Math 1497 - Calc 2

Infinite Series

$$\sum_{n=1}^{\infty} a_n$$

Special Series

(1) Geometric $a + ar + ar^2 + \dots$ converges if $|r| < 1$

if so $S_\infty = \frac{a}{1-r}$

(2) Telescopic

$$\sum a_n = \sum b_n - b_{n+1}$$

(i) stop at N, (ii) write out terms and cancel

(iii) let $N \to \infty$

(3) Harmonic $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ diverges

(4) p series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $p > 1$ converges

$p < 1$ diverges

Tests (1) n^{th} term

If $\lim_{n\to\infty} a_n = \#$ ($\neq 0$) the series diverges

if $\lim_{n\to\infty} a_n = 0$ then nothing can be said!

Ex 1 $\sum_{n=1}^{\infty} \frac{2n^3+n-2}{5n^3-n+1}$

$$\lim_{n\to\infty} \frac{2n^3+n-2}{5n^3+n+1} = \frac{2}{5} \neq 0$$

so by n^{th} term test the series div

Test 2 $\int$ test

If $a_n = f(n)$

Conditions (i) $f \geq 0$ (ii) f cont^s (iii) f dec $f' < 0$

if $\int_1^{\infty} f(x)dx$ converges (diverges) $\sum_{n=1}^{\infty} a_n$ converges (diverges)

Ex 2 $\sum_{n=1}^{\infty} \frac{e^{-n}}{e^{-n}+1}$

$f(x) = \frac{e^{-x}}{e^{-x}+1}$

(1) $f \geq 0$ ✓

(2) f cont^s ✓

$$f' = \frac{-e^{-x}(e^{-x}+1) + e^{-x}(e^{-x})}{(e^{-x}+1)^2} = \frac{-e^{-x}}{(e^{-x}+1)^2} < 0$$

so test applies!

$$\int_1^{\infty} \frac{e^{-x}}{e^{-x}+1} dx = \lim_{b\to\infty} \int_1^b \frac{e^{-x}}{e^{-x}+1} dx$$

$$= \lim_{b\to\infty} -\ln|e^{-x}+1| \Big|_1^b = \lim_{b\to\infty} -\ln(e^{-b}+1) + \ln(e^{-1}+1)$$

$$= \ln(e^{-1}+1)$$

so by $\int$ test the series conv.

Test #3 LCT we compare 2 series

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \#$ (not zero)

the 2 series either conv or div.

Ex 3 $\sum_{n=4}^{\infty} \frac{2}{n^2-10}$ compare w/ $\sum_{n=4}^{\infty} \frac{1}{n^2}$ $p=2$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2-10}} = \lim_{n\to\infty} \frac{n^2-10}{2n^2} = \frac{1}{2} \; (\neq 0)$$

$\because \sum \frac{1}{n^2}$ conv then our series converges

Test #4 DCT

if $0 < a_n \le b_n$

if $\sum a_n$ div $\sum b_n$ div

if $\sum b_n$ conv $\sum a_n$ conv

Ex $\sum \frac{2+\sin n}{n}$ $\because -1 \le \sin n$ then $-1+2 \le 2+\sin n$

$\Rightarrow \frac{1}{n} \le \frac{2+\sin n}{n}$ $\because \sum \frac{1}{n}$ div

then our series div by DCT.

Test 5 Ratio Test

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$, if $L<1$ series conv

$L>1$ series div

$L=1$ no conclusion

ex 5 $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2n}} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \cdot \frac{n+1}{n} = \lim_{n\to\infty} \frac{\cancel{2^n}}{2\cancel{2^n}} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$$ so by ratio test the series conv

Test 6 Root Test

If $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = L$

$L>1$ div

$L<1$ conv

$L=1$ no conclusion

ex 6 $\sum\limits_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$ $\lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e > 1$

So by root test, the series div

Up to now, all of the terms in the series have been positive. We now consider when some are negative. For example

$$1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots$$

The possibilities are endless. So we will consider a special type of series when the terms alternate sign. For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

the $(-1)^{n+1}$ is $=1$ for n odd & -1 when n even so this alternates the sign

We know that $\sum \frac{1}{n}$ diverges but what about the alternating harmonic series

Let add the terms in two's

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right)$$

$$\frac{2-1}{2 \cdot 1} + \frac{4-3}{4 \cdot 3} + \frac{6-5}{6 \cdot 5} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 2n}$$

LCT w/ $\sum \frac{1}{n^2}$ $p=2$ shows it converges.

Test 7 Alternating Series Test (AST)

Given the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ $\quad (a_n \geq 0)$

this series converges if

(i) a_n is decreasing $(a_{n+1} \leq a_n)$

(ii) $\lim_{n \to \infty} a_n = 0$

Previous Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

(i) $f(x) = \frac{1}{x}$ $\quad f' = -\frac{1}{x^2} < 1$ so decreasing ✓

(ii) $\lim_{n \to \infty} \frac{1}{n} = 0$ ✓

So series div by AST

$\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$ we conclude write this as

$\sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$ which is geometric w/ $r = -2/3$

so it conv.

OR $\sum (-1)^n \left(\frac{2}{3}\right)^n$

(1) decreasing $a_n = \left(\frac{2}{3}\right)^n \quad a_{n+1} = \left(\frac{2}{3}\right)^{n+1}$

$a_{n+1} \overset{?}{<} a_n$

$\left(\frac{2}{3}\right)^{n+1} \overset{?}{<} \left(\frac{2}{3}\right)^n \qquad \frac{2}{3}\left(\frac{2}{3}\right)^n \overset{?}{<} \left(\frac{2}{3}\right)^n$

is $\frac{2}{3} < 1$ ✓ so yes dec

(2) $\lim_{n \to \infty} \left(\frac{2}{3}\right)^n \to 0$ so by AST the series conv

Ex $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$

(1) dec? $f(x) = \frac{\ln x}{x} \quad f' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2}$

$= \frac{1 - \ln x}{x^2} < 0$ if $x > e$

2.7.

dc

(2) $\lim_{n \to \infty} \frac{\ln n}{n}$

$\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$

So conv by AST.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

(i) dec. ?

ii) $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$

no need to check this 1st condition

Consider the following 2 series

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - - -$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - -$

these both conv by AST

However if we stop the series from alternating

$\sum \frac{1}{n}$ div $\sum \frac{1}{n^2}$ conv.

we want to disquish between the convergence

as $\sum \frac{(-1)^n}{n}$ converges very slowly

$\sum \frac{(-1)^n}{n^2}$ conv. very fast,

Absolute & Conditional Convergence

if $\sum_{n=1}^{\infty} a_n$ div but $\sum_{n=1}^{\infty} (-1)^n a_n$ conv

then we say it conv. conditionally

if $\sum_{n=1}^{\infty} a_n$ conv & $\sum (-1)^n a_n$ conv

we say converges absolutely

Note: if we have $\sum_{n=1}^{\infty} a_n (-1)^n$

check $\sum a_n$ 1st - if it converges we done

if not - check via AST.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ 1st check $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ it conv

($\int$ test, LCT, DCT)

so conv absolutely

ex $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ 1st check $\sum \frac{n}{n^2+1}$ div LCT $\sum \frac{1}{n}$

by AST converges conditionally

(Bonus Question)