Calculus 3 - Limits

In the previous class we considered

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = \frac{``0"}{"0"} \tag{1}$$

Along $x = 0$, $y = 0$ and $y = x$ we get the limit is zero. So maybe the limit is actually zero. The way that we'll show this (or prove this) is using the squeeze theorem.

Squeeze Theorem

If

$$f \leq g \leq h \tag{2}$$

near $x = a$ and if

$$\lim_{x\to a} f = L, \quad \lim_{x\to a} h = L, \tag{3}$$

then

$$\lim_{x\to a} g = L. \tag{4}$$

This also works for limits involving more that one independent variable.

We first start with some basic inequalities.

$$\begin{aligned} -\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2}, \\ -\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}. \end{aligned} \tag{5}$$

The idea is that we manipulate these to to eventually get to the function we're trying to take the limit of. So, in the first example, we want to get to

$$\frac{x^2y^2}{x^2+y^2} \tag{6}$$

First

$$
\begin{aligned}
-\left(\sqrt{x^2+y^2}\right)^2 \leq x^2 \leq \left(\sqrt{x^2+y^2}\right)^2, \\
-\left(\sqrt{x^2+y^2}\right)^2 \leq y^2 \leq \left(\sqrt{x^2+y^2}\right)^2.
\end{aligned} \tag{7}
$$

Then multiply these together

$$
-\left(\sqrt{x^2+y^2}\right)^4 \leq x^2y^2 \leq \left(\sqrt{x^2+y^2}\right)^4, \tag{8}
$$

or

$$
-\left(x^2+y^2\right)^2 \leq x^2y^2 \leq \left(x^2+y^2\right)^2, \tag{9}
$$

Then divide by x^2+y^2

$$
-\frac{\left(x^2+y^2\right)^2}{x^2+y^2} \leq \frac{x^2y^2}{x^2+y^2} \leq \frac{\left(x^2+y^2\right)^2}{x^2+y^2} \tag{10}
$$

or

$$
-\left(x^2+y^2\right) \leq \frac{x^2y^2}{x^2+y^2} \leq \left(x^2+y^2\right) \tag{11}
$$

so

$$
\lim_{(x,y)\to(0,0)} -\left(x^2+y^2\right) \leq \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \leq \lim_{(x,y)\to(0,0)} \left(x^2+y^2\right) \tag{12}
$$

Since

$$\lim_{(x,y)\to(0,0)} x^2 + y^2 = 0 \tag{13}$$

then by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0. \tag{14}$$

Example 2

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 2y^4}{x^2 + y^2} \tag{15}$$

We start with the basic inequalities

$$\begin{aligned} -\sqrt{x^2 + y^2} \le x \le \sqrt{x^2 + y^2}, \\ -\sqrt{x^2 + y^2} \le y \le \sqrt{x^2 + y^2}. \end{aligned} \tag{16}$$

so

$$\begin{aligned} -\left(x^2 + y^2\right)^2 \le x^4 \le \left(x^2 + y^2\right)^2, \\ -\left(x^2 + y^2\right)^2 \le y^4 \le \left(x^2 + y^2\right)^2. \end{aligned} \tag{17}$$

so

$$-3\left(x^2 + y^2\right)^2 \le x^4 + 2y^4 \le 3\left(x^2 + y^2\right)^2 \tag{18}$$

so

$$-3\left(x^2 + y^2\right) \le \frac{x^4 + 2y^4}{x^2 + y^2} \le 3\left(x^2 + y^2\right) \tag{19}$$

Since

$$\lim_{(x,y)\to(0,0)} x^2 + y^2 = 0 \tag{20}$$

then by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 2y^4}{x^2 + y^2} = 0. \tag{21}$$

Example 3

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^2} \tag{22}$$

We start with the basic inequalities

$$\begin{aligned} -\sqrt{x^2 + y^2} \leq x \leq \sqrt{x^2 + y^2}, \\ -\sqrt{x^2 + y^2} \leq y \leq \sqrt{x^2 + y^2}. \end{aligned} \tag{23}$$

so

$$\begin{aligned} -\sqrt{x^2 + y^2} \leq x \leq \sqrt{x^2 + y^2}, \\ -\left(x^2 + y^2\right)^{3/2} \leq y^3 \leq \left(x^2 + y^2\right)^{3.2}. \end{aligned} \tag{24}$$

so

$$-\left(x^2 + y^2\right)^2 \leq xy^3 \leq \left(x^2 + y^2\right)^2 \tag{25}$$

so

$$-\left(x^2 + y^2\right) \leq \frac{xy^3}{x^2 + y^2} \leq \left(x^2 + y^2\right) \tag{26}$$

Since

$$\lim_{(x,y)\to(0,0)} x^2 + y^2 = 0 \tag{27}$$

then by the squeeze theorem

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^2} = 0. \tag{28}$$

So a question you might have is: When do think a limit will exist or will not exist. Let consider a few

$$(i) \quad \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \quad DNE \qquad (ii) \quad \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} \quad DNE$$

$$(iii) \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} \quad Exists \qquad (iv) \quad \lim_{(x,y)\to(0,0)} \frac{x^4+2y^4}{x^2+y^2} \quad Exists$$

We also have

$$(v) \quad \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2} \quad DNE \qquad (vi) \quad \lim_{(x,y)\to(0,0)} \frac{x^4+2x^2y^2+y^4}{x^2+y^2} \quad Exists$$

We look at powers (and combined powers).
(i) power in numerator is 2 and 2, in denominator 2 and 2. Powers are the same so limit DNE
(ii) combined power in numerator is 2, in denominator 2 and 2. Powers are the same so limit DNE
(iii) combined power in numerator is 4, in denominator 2 and 2. Power

in numerator is higher than that of the denominator so the limit exists (although you will need to show this).
(iv) powers in numerator is 4 and 4. Powers in numerator is higher than that of the denominator so the limit exists (again, you will need to show this).
(v) power in numerator is 1 and 1, in denominator 2 and 2. Powers are lower in the numerator than the denominator so the limit DNE (follow different paths)
(vi) powers in numerator is 4, 4 and 4. In the denominator is 2 and 3. Powers in numerator is higher than that of the denominator so the limit exists.
Example 4

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^4}{x^2 + y^2} \tag{29}$$

This problem is interesting since in the numerator we have a power 2 and a power of 4. If we split the limit up into

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2} + \lim_{(x,y)\to(0,0)} \frac{y^4}{x^2 + y^2} \tag{30}$$

The first limit DNE whereas the second does so overall, the limit DNE!
Example 5

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + 2y^2} \tag{31}$$

Using the powers we see that this limit should exist but how do we use the inequalities given in (34)?

Well, we just need to manipulate them a little more. Remember, the goal is to try and get an inequality where

$$\frac{x^2y^2}{x^2+2y^2} \tag{32}$$

is in the center. So

$$\begin{aligned} -\sqrt{x^2+2y^2} \le -\sqrt{x^2+y^2} \le x \le \sqrt{x^2+y^2} \le \sqrt{x^2+2y^2}, \\ -\sqrt{x^2+2y^2} \le -\sqrt{x^2+y^2} \le y \le \sqrt{x^2+y^2} \le \sqrt{x^2+2y^2}. \end{aligned} \tag{33}$$

or

$$\begin{aligned} -\sqrt{x^2+2y^2} \le x \le \sqrt{x^2+2y^2}, \\ -\sqrt{x^2+2y^2} \le y \le \sqrt{x^2+2y^2}. \end{aligned} \tag{34}$$

The rest follows like what we did in the first three examples.

Continuity

A function $f(x,y)$ is continuous at a point (a,b) if $f(a,b)$ is defined, the limit of $f(x,y)$ exists at (a,b) and

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \tag{35}$$

The function $f(x,y)$ is said to be continuous in the open region R if it is continuous at every point in R.