# Math 6345 AODE's

Consider

$$\frac{d\bar{x}}{dt} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \bar{x}$$

so using linear algebra

$$\begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-2) = 0$$

$$\lambda = 1, 2$$

So $\lambda = 1$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \bar{e} = \bar{0} \Rightarrow \bar{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \bar{x}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t}$$

$\lambda = 2$

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \bar{e} = \bar{0} \Rightarrow \bar{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ so } \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$\bar{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

$$= \begin{pmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix} \bar{c} \quad \text{where } (\;) = \Phi(t)$$

fundamental matrix

Now suppose we ~~had~~ $\bar{x}(0) = \vec{x}_0$, some IC

then $\bar{x}(0) = \Phi(0)\,\bar{c} = \bar{x}_0 \Rightarrow \bar{c} = \Phi^{-1}(0)$

so $\Phi(0) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$\det \Phi(0) = -1$

so $\Phi^{-1}(0) = -\frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

so $\bar{x} = \Phi(t)\,\Phi^{-1}(0)\,\bar{x}_0$

$= \begin{pmatrix} 0 & e^{2t} \\ e^{t} & e^{2t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\bar{x}_0$

$= \begin{pmatrix} e^{2t} & 0 \\ e^{2t} - e^{t} & e^{t} \end{pmatrix}\bar{x}_0$

Last class,

$$\frac{dx}{dt} = ax \Rightarrow x = ce^{at}$$

$$\frac{d\bar{x}}{dt} = A\bar{x} \Rightarrow \bar{x} = e^{At}\bar{c}$$

Now if $\bar{x}(0) = \vec{x}_0$

$$e^{[0]}\bar{c} = \bar{x}_0$$

suppose $e^{[0]} = I$ - identity

then $I\bar{c} = \bar{x}_0 \Rightarrow \bar{c} = \bar{x}_0$

so $\bar{x} = e^{At}\bar{x}_0$

$$= \Phi(t)\,\Phi^{-1}(0)\,\bar{x}_0$$

$$\Rightarrow \boxed{\Phi(t)\,\Phi^{-1}(0) = e^{At}}$$

But still, what is $e^{At}$?

Def$^{n}$ Let $A$ be a real valued $n \times n$ matrix

then $$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

note $$e^{[0]} = I$$

we will want to question on whether this converges (later)!

Lets see if this leads to the sol$^{n}$ in the last case

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 7 & 1 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 15 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}$$ prove this!

so $e^{At} = \begin{pmatrix}1&0\\0&1\end{pmatrix} + \begin{pmatrix}2&0\\1&1\end{pmatrix}t + \begin{pmatrix}4&0\\3&1\end{pmatrix}\frac{t^2}{2!} + \cdots$

$$= \begin{pmatrix} 1+2t+\frac{4t^2}{2!}+\frac{8t^3}{3!} & 0 \\ t+\frac{3t^2}{2!}+\frac{7t^3}{3!}+\cdots & 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots \end{pmatrix}$$

Now $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ (Calc 2)

so $1+2t+\frac{(2t)^2}{2!}+\frac{(2t)^3}{3!}+\cdots \quad e^{2t}$

What about

$t+\frac{3t^2}{2!}+\frac{7t^3}{3!}+\cdots$ ? (bottom left corner of $e^{At}$)

From Sol$^n$

$$e^{2t}-e^{t} = t+\frac{3t^2}{2!}+\frac{7t^3}{3!}+\cdots$$

but this might have been hard to deduce

Ex $\dfrac{d\bar{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \bar{x}$

Linear Algebra

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad 4-4\lambda+\lambda^2-1=0$$

$$(\lambda-1)(\lambda-3)=0$$

$$\lambda = 1, 3$$

$\lambda = 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \bar{e} = \bar{0} \qquad \bar{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \bar{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$$

$\lambda = 3$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \bar{e} = \bar{0} \Rightarrow \bar{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$$

$$\bar{x} = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \vec{c}$$

$$\Phi(t) = \begin{pmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \det \Phi(0) = 2$$

$$\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Phi(t)\,\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} e^{t} & e^{3t} \\ -e^{t} & e^{3t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{t}+e^{3t} & -e^{t}+e^{3t} \\ -e^{t}+e^{3t} & e^{t}+e^{3t} \end{pmatrix}$$

Now via $e^{At}$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

so $e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t + \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \frac{t^3}{3!} + \cdots$

$$= \begin{pmatrix} 1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} + \cdots & t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \cdots \\ t + \frac{4t^2}{2!} + \frac{13t^3}{3!} + \cdots & 1 + 2t + \frac{5t^2}{2!} + \frac{14t^3}{3!} \end{pmatrix}$$

and this would have been hard to deduce closed form soln's!

However note

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so $e^{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} t} = e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} t}$

$= e^{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} t} \cdot e^{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$ could this be useful?