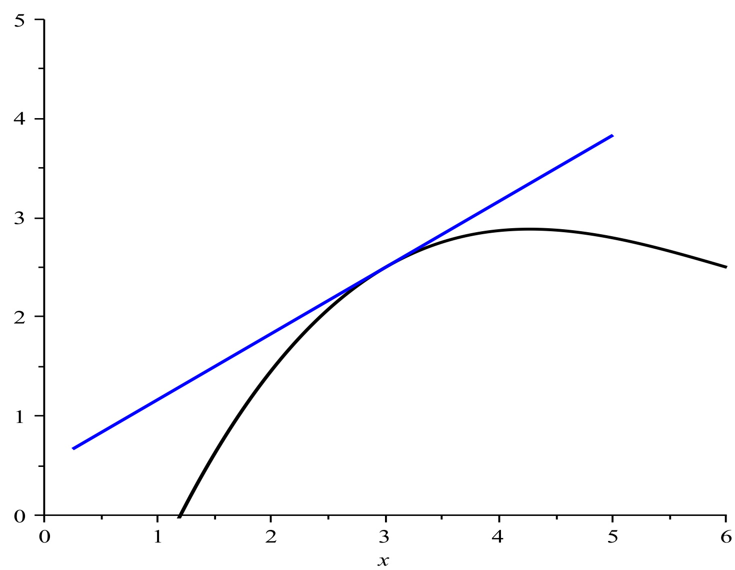
Calculus 3 - Differentials

In this class we derive the differential for functions of more than one variable. Recall from Calc 1, the differential for a function $y = f(x)$

$$dy = f'(x)dx. \tag{1}$$

So if $y = x^2$ then $dy = 2xdx$. If $y = \tan x$, $dy = \sec^2 xdx$. To get an idea where this (the differential) came from let us consider the tangent problem. The equation of the tangent for $y = f(x)$ at some point $(a, f(a))$is

$$y - f(a) = f'(a)(x - a) \tag{2}$$



Suppose we start at the point $(a, f(a))$ and move a little bit along the curve, say from $x = a$ to $x = a + \Delta x$. Then the actual change in y would be

$$\Delta y = f(a + \Delta x) - f(a) \tag{3}$$

Now let us move along the tangent line and let the small changes in x and

y be dx and dy so

$$x = a + dx, \quad y = f(a) + dy \tag{4}$$

Using the tangent line (2) we obtain

$$f(a) + dy - f(a) = f'(a)(a + dx - a) \tag{5}$$

and after cancellation we obtain

$$dy = f'(a)dx \tag{6}$$

From we define the differential that is given in (1).

Differentials in Higher Dimensions

So how do we extend differentials to functions of more than one variable? Let us consider functions of two variables and $z = f(x, y)$. As we did in 2D following the tangent line, in 3d, we follow the tangent plane. Recall, for $z = f(x, y)$ and the point (a, b, c) where $c = f(a, b)$, the tangent plane is

$$f_x(a,b)(x - a) + f_y(a,b)(y - b) - (z - c) = 0 \tag{7}$$

We move a small amount (dx, dy, dz) from the point (a, b, c) so

$$x = a + dx, \quad y = b + dy, \quad x = c + dz \tag{8}$$

This we substitute into (17) so

$$f_x(a,b)(a + dx - a) + f_y(a,b)(b + dy - b) - (c + dz - c) = 0 \tag{9}$$

and after cancellation, we obtain

$$dz = f_x(a,b)dx + f_y(a,b)dy \tag{10}$$

as as we did in 2D, we define the differential in 3D as

$$dz = f_x dx + f_y dy \tag{11}$$

or

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \tag{12}$$

(this latter form we will need tomorrow).

Let us look at some examples.

Example 1.

If $z = x^2 + y^2$ find dz

Soln.

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \tag{13}$$

so

$$dz = 2x\, dx + 2y\, dy \tag{14}$$

Example 2.

If $z = 2x^3y - 8xy^4$ find dz

Soln.

$$\frac{\partial f}{\partial x} = 6x^2y - 8y^4, \quad \frac{\partial f}{\partial y} = 2x^3 - 32xy^3, \tag{15}$$

so

$$dz = \left(6x^2y - 8y^4\right) dx + \left(2x^3 - 32xy^3\right) dy \tag{16}$$

Differential in More Variables

Differentials easily extends to more variables. So if $w = f(x, y, z)$ then

$$dw = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz \tag{17}$$

Example 3.
If $w = \dfrac{x+y}{z-3y}$ find dw
Soln.

$$\frac{\partial w}{\partial x} = \frac{1}{z-3y}, \quad \frac{\partial w}{\partial y} = \frac{1(z-3y) - (x+y)(-3)}{(z-3y)^2}, \quad \frac{\partial f}{\partial z} = -\frac{x+y}{(z-3y)^2}, \tag{18}$$

so

$$dz = \frac{(z-3y)dx + (3x+z)dy - (x+y)dz}{(z-3y)^2} \tag{19}$$

Applications

One nice feature of differentials is that they can be using to approximate change. The following examples illustrates this.

Example 4. Pg 909, # 18
Approximate $\sqrt{4.03^2 + 3.1^2} - \sqrt{4^2 + 3^2}$
Soln.
First, the exact value is 0.084378035. Next we will use differentials. We define

$$f = \sqrt{x^2 + y^2} \tag{20}$$

and so

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$
$$= \frac{x}{\sqrt{x^2+y^2}}\, dx + \frac{y}{\sqrt{x^2+y^2}}\, dy \tag{21}$$

Now we go from $(4, 3)$ to $(4.03, 3.1)$ so the change is $dx = 0.03$ and $dy = 0.1$ so

$$df = \frac{4}{\sqrt{4^2+3^2}}\, 0.03 + \frac{3}{\sqrt{4^2+3^2}}\, 0.1 = \frac{4}{5}\, 0.03 + \frac{3}{5}\, 0.1 = 0.084000 \tag{22}$$

which gives a good approximation to the actual answer of 0.084378035.

Example 5.

Suppose we were to construct a box that measured $6'' \times 6'' \times 4''$. If instead we measured $5.95'' \times 6.1'' \times 3.92''$, approximate the error in volume.

Soln.

Here we define $V = xyz$ and calculate

$$dV = yz\, dx + xz\, dy + xy\, dz \tag{23}$$

Next, we use the values $(6, 6, 4)$ and errors $(-0.05, 0.1, -0.08)$. From (24) we obtain

$$dV = 6 \cdot 4 \cdot (-0.05) + 6 \cdot 4 \cdot 0.1 + 6 \cdot 6 \cdot (-0.08) = -1.68 \tag{24}$$

The actual error is -1.7236. A relative error of 1.2%.