Getting a Wave Equation from Maxwell's Equations

In this writing I will write electric field quantities in red text, and magnetic field quantities in blue for clarity. The wave equation for the electric field will be found for **source-free** regions first. This yields a homogeneous wave equation (an equation that equals zero). The magnetic one is a similar derivation and will just be stated. Next the inhomogeneous wave equation will be derived which comes from regions that have **sources and currents present**. This will be done in terms of the magnetic vector potential and electric scalar potential, rather than the fields themselves.

Homogeneous Wave Equation (Source-Free Regions)

Let's look first at Maxwell's equations in differential form for **source-free regions** (i.e. no charges or currents in the region).

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3)$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad (4)$$

Note that the coloring scheme allows us to see by inspection that time derivatives of one type of field lead to the other.

Now let us look at the most basic form of a wave equation:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

This says if the second spatial derivative equals the second time derivative (multiplied by a constant), the function ψ describes a wave. This equation is for 1 dimension, like a wave traveling down a string that's vibrating up and down. **Note:** c in this equation represents the speed of propagation of the wave.

So how can Maxwell's Equations be put into this form? Doing so will require a very useful vector identity that will be introduced shortly. Here's how it's done:

Consider first Eq. 3, Faraday's Law:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

In words this says that a time change in a magnetic flux density is accompanied by an electric field that "curls around" the line pointing along the direction in which the magnetic field *decreases*. That negative sign (called Lenz's Law) is why we say it's along the decreasing direction.

Now let's take the curl of both sides of Faraday's Law:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} \qquad (5)$$

Now we introduce the vector identity mentioned earlier. For any vector $\vec{A}$,

$$\vec{\nabla}\times\vec{\nabla}\times A = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right) - \nabla^2\vec{A}$$

In words this says that the curl of the curl of a vector equals the gradient of the divergence of the vector minus the Laplacian of the vector. The vector Laplacian is basically the second spatial derivative. Applying this to Eq. (5) gives:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} \qquad (6)$$

At this point let us recall the two constitutive relationships:

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{B} = \mu_0 \vec{H}$$

By these relationships, we could rewrite Eq. 6 as

$$\frac{1}{\varepsilon_0}\vec{\nabla}(\vec{\nabla} \cdot \vec{D}) - \nabla^2 \vec{E} = -\mu_0 \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}$$

Now we can see that the first term contains a divergence of electric flux density, $\vec{\nabla} \cdot \vec{D}$ and by Eq. 1 we know this is zero for source-free regions. Thus:

$$-\nabla^2 \vec{E} = -\mu_0 \vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}$$

Now let's look at the right-hand side of this equation. The curl operator "$\vec{\nabla} \times$" is a spatial derivative. The time derivative and the spatial derivative can be taken in any order since they are linear operators acting on different variables (i.e. the partial time derivative sees the spatial coordinates as constants). Thus we can rearrange it to state

$$-\nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

Now we can substitute the curl of the magnetic field according to Eq. 4.

$$-\nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{D}}{\partial t}\right)$$

Using the first constitutive relationship, $\vec{D} = \varepsilon_0\vec{E}$, and combining the two time derivatives, this can be written

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Now moving the right side over to the left and rearranging gives

$$\nabla^2\vec{E} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Since the Laplacian operator, ∇^2, is a second spatial derivative, we can now state (with the continued assumption our wave is along only the *x*-axis of our chosen reference frame):

$$\frac{\partial^2\vec{E}}{\partial x^2} - \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Which is a 1-D wave equation in $\vec{E}$. Recalling that the coefficient on the second time derivative is $\frac{1}{c^2}$, where c is the speed of the wave, we can see the wave travels at a speed of

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299{,}792{,}458\, m/s$$

which is the known speed of light.
The same relationship can be worked out for the magnetic field resulting in

$$\frac{\partial^2 \vec{H}}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Notice that Maxwell's equations are first-order differential equations that are coupled. That is, Eqs. 3 & 4 have only first order differentials (curls) but have both electric and magnetic quantities. These wave equations we've just derived are now uncoupled but are done so at the expense of raising them to second-order differential equations.

Inhomogeneous Wave Equation (Sources Present)

Now let's look at Maxwell's equations in differential form for regions containing sources.

$$\vec{\nabla} \cdot \vec{D} = \rho_v \qquad (7)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (8)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (9)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (10)$$

Note that Eq. 7 is now equated to an electric volume charge density, ρ_v, and Eq. 10 has an additional term on the right side, $\vec{J}$, the current density. These will result in inhomogeneous wave equations.

Now we will derive the electric scalar potential V and magnetic vector potential, $\vec{A}$. The final wave equation will be in terms of these, not the fields themselves.

Magnetic Vector Potential $\vec{A}$

With Eq. 8, we can see that the divergence of the magnetic flux density is always zero. A mathematical identity states that

$$\nabla \cdot \left(\vec{\nabla}\times\vec{F}\right) = 0$$

That is, the divergence of the curl is always zero. Since the divergence of $\vec{B}$ is zero in Eq. 8, we can let $\vec{B}$ equal the curl of some other vector called the **magnetic vector potential**, $\vec{A}$.

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

Electric Scalar Potential V

Now Eq. 8 *must* be zero because it is the divergence of a curl. Next let's insert this result into Eq. 9

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{\nabla}\times\vec{A}}{\partial t}$$

Since these are both curls, which are linear operators, we can collect terms

$$\vec{\nabla}\times\left(\vec{E}+\frac{\partial\vec{A}}{\partial t}\right)=0 \qquad (11)$$

Another useful mathematical identity can now be employed, namely that

$$\vec{\nabla}\times(-\vec{\nabla}V)=0$$

The curl of a (negative) gradient is identically zero. Equation 11 is in exactly this form so we may set the operand of the curl operator as a negative gradient of another *scalar* quantity called the **electric scalar potential**, V (i.e. voltage)

$$\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}V$$

Or

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \qquad (12)$$

If the situation is static, Eq. 12 becomes the familiar $\vec{E}=-\nabla V$.

Inhomogeneous Wave Equation in $\vec{A}$

If we now consider Eq. 10 and substitute the constitutive relations for homogeneous media we have

$$\vec{\nabla}\times\vec{B}=\mu\vec{J}+\mu\varepsilon\frac{\partial\vec{E}}{\partial t}$$

If we now use the relations $\vec{B}=\vec{\nabla}\times\vec{A}$ and $\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$, we get

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \mu\vec{J} + \mu\varepsilon\frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$$

Employing the relation $\vec{\nabla}\times\vec{\nabla}\times A = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$, we get

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A} = \mu\vec{J} - \mu\varepsilon\vec{\nabla}\frac{\partial V}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2}$$

Rearranging gives

$$\vec{\nabla}^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t}\right) \qquad (13)$$

At this point we introduce one more relationship called the Lorenz gauge. This is required to give a unique vector potential $\vec{A}$, and is done by specifying its divergence. This additional degree of freedom can be defined to simplify the right side of Eq. 13. The Lorenz gauge is defined as

$$\vec{\nabla}\cdot\vec{A} + \mu\varepsilon\frac{\partial V}{\partial t} = 0 \qquad (14)$$

So that

$$\vec{\nabla}^2\vec{A} - \mu\varepsilon\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J} \qquad (15)$$

Which is indeed an inhomogeneous wave equation in the magnetic vector potential.

Inhomogeneous Wave Equation in V

Using Eq. 7 and $\vec{D} = \varepsilon\vec{E}$ we see that $\vec{\nabla} \cdot \vec{E} = \rho_v/\varepsilon$. Using Eq. 12 we can then state

$$\vec{\nabla} \cdot \left(\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t}\right) = -\frac{\rho_v}{\varepsilon}$$

Distributing the vector operator on the left hand side gives

$$\vec{\nabla}^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\frac{\rho_v}{\varepsilon} \qquad (16)$$

If we solve the Lorenz gauge of Eq. 14 for an alternate expression for $\vec{\nabla} \cdot \vec{A}$, we get

$$\vec{\nabla} \cdot \vec{A} = -\mu\varepsilon\frac{\partial V}{\partial t}$$

Putting this into Eq. 16 gives

$$\vec{\nabla}^2 V + \frac{\partial}{\partial t}\left(-\mu\varepsilon\frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\varepsilon}$$

Or

$$\vec{\nabla}^2 V - \mu\varepsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \qquad (17)$$

Which is an inhomogeneous wave equation in the electric scalar potential.