

Calculus 3 - Cylindrical and Spherical Coordinates

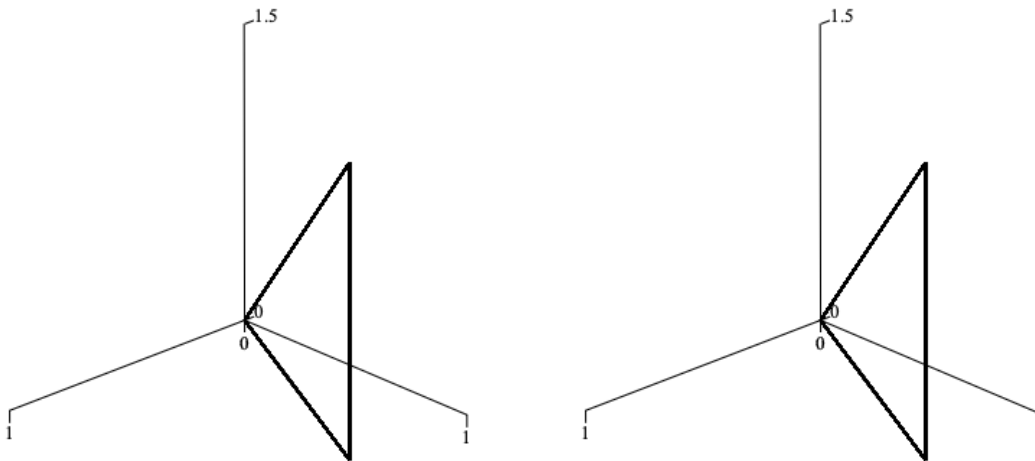
Cylindrical Polar Coordinates

These are similar to the usual polar coordinates that are in 2D but we simply add z to extend to 3D so

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. \quad (1)$$

Spherical Polar Coordinates

$$x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi. \quad (2)$$



Triple Integrals

We consider triple integrals

$$V = \iiint_V F(x, y, z) dV \quad (3)$$

in each of these two coordinate systems. We first consider cylindrical and then spherical.

Cylindrical Polar Coordinates

This integral has three main parts:

1. the integrand
2. dV
3. limits

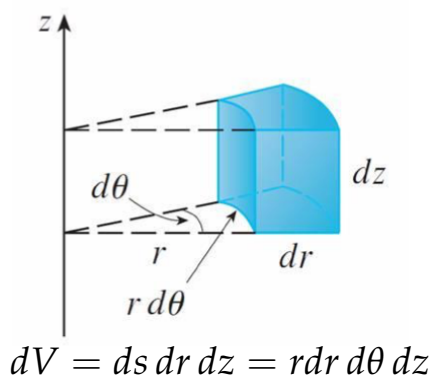
1. *The integrand.* For this part, we simply substitute in

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \quad (4)$$

into $F(x, y, z)$ and simplify. So, in general,

$$\iiint_V F(x, y, z) dV = \iiint_V F(r \cos \theta, r \sin \theta, z) dV. \quad (5)$$

2. dV . In Cartesian coordinates, this is $dV = dx dy dz$. In cylindrical coords



$$dV = ds dr dz = r dr d\theta dz \quad (6)$$

3. *Limits of Integration.* These ultimately come from the picture of the volume itself. So

$$\int_{\alpha}^{\beta} \int_{r_i(\theta)}^{r_o(\theta)} \int_{f_1(r, \theta)}^{f_2(r, \theta)} F(r \cos \theta, r \sin \theta, z) r dz dr d\theta \quad (7)$$

Spherical Polar Coordinates

This integral has three main parts:

$$\iiint_V F(x, y, z) dV \quad (8)$$

1. the integrand
2. dV
3. limits

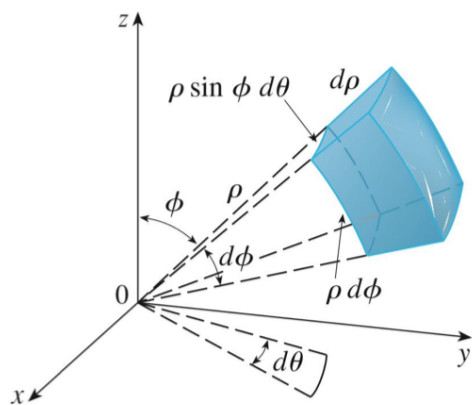
1. *The integrand.* For this part, we simply substitute in

$$x = \rho \cos \theta \sin \phi, \quad y = \rho \sin \theta \sin \phi, \quad z = \rho \cos \phi. \quad (9)$$

into $F(x, y, z)$ and simplify. So, in general,

$$\iiint_V F(x, y, z) dV = \iiint_V F(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) dV. \quad (10)$$

2. dV . In Cartesian coordinates, this is $dV = dx dy dz$. In spherical coords



$$dV = ds_1 ds_2 d\rho = \rho^2 \sin \phi d\rho d\phi d\theta \quad (11)$$

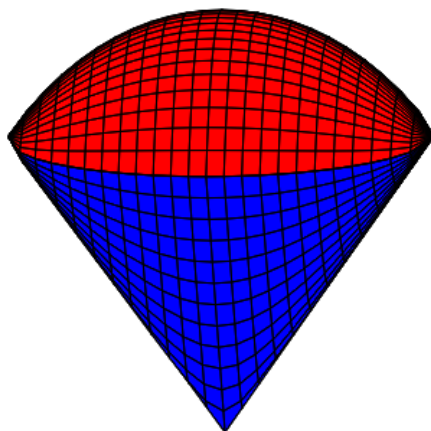
3. *Limits of Integration.* These ultimately come from the picture of the volume itself. So

$$\int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} F(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta. \quad (12)$$

Example 1. Set up the triple integral for the volume bound by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$ and evaluate

$$\iiint_V z dV \quad (13)$$

Soln:



We first convert each surface to spherical polar coords. They become $\rho = \sqrt{2}$ and $\phi = \pi/4$. We see that

$$\rho = 0 \rightarrow \sqrt{2}, \quad \phi = 0 \rightarrow \pi/4, \quad \text{and} \quad \theta = 0 \rightarrow 2\pi.$$

$$\begin{aligned}
& \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho \cos \phi \rho^2 \sin \phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \left. \frac{1}{4} \rho^4 \right|_0^{\sqrt{2}} \sin \phi \cos \phi d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \sin \phi \cos \phi d\phi d\theta \\
&= \int_0^{2\pi} \left. \frac{1}{2} \sin^2 \phi \right|_0^{\pi/4} d\theta \\
&= \frac{1}{4} \int_0^{2\pi} d\theta \\
&= \frac{1}{4} \theta \Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned} \tag{14}$$

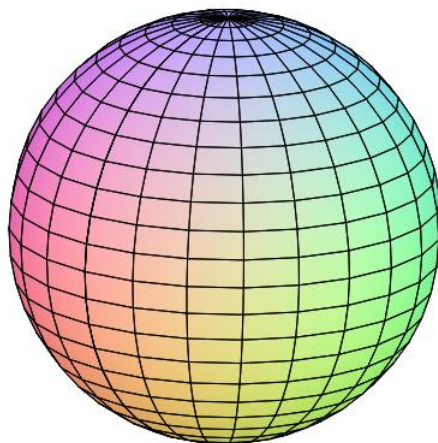
Let us compare with the same setup in cylindrical polar coordinates.

$$\begin{aligned}
& \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} z r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 r \left. \frac{1}{2} z^2 \right|_r^{\sqrt{2-r^2}} dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^1 r (2 - r^2 - r^2) dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \int_0^1 (2r - 2r^3) dr d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left. r^2 - \frac{1}{2} r^4 \right|_0^1 d\theta \\
&= \frac{1}{4} \theta \Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned} \tag{15}$$

Example 2. Set up the triple integral for the volume bound by the sphere $x^2 + y^2 + z^2 = 1$ and evaluate

$$\iiint_V 1 dV \quad (16)$$

Soln:



We first convert each surface to spherical polar coords. They become $\rho = 1$.

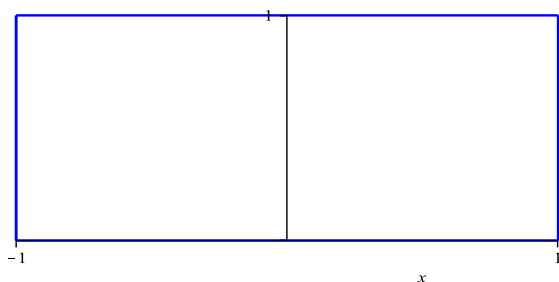
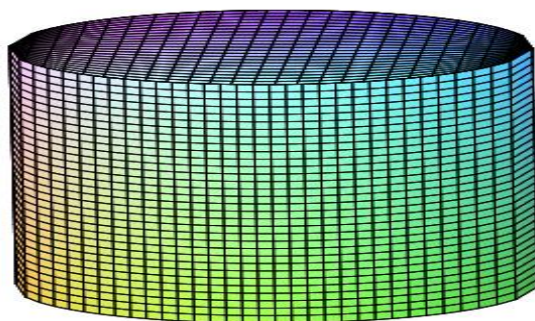
We see that $\rho = 0 \rightarrow 1$, $\phi = 0 \rightarrow \pi$, and $\theta = 0 \rightarrow 2\pi$.

$$\begin{aligned} & \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_0^\pi \frac{1}{3} \rho^3 \Big|_0^1 \sin \phi d\phi d\theta \\ &= \frac{1}{3} \int_0^{2\pi} -\cos \phi \Big|_0^\pi d\phi d\theta \\ &= \frac{1}{3} \times 2 \int_0^{2\pi} d\theta \\ &= \frac{1}{3} \times 2 \times 2\pi = \frac{4}{3}\pi \end{aligned} \quad (17)$$

Example 3. Set up the triple integral for the volume bound by $z = 0, z = 1$, and inside the cylinder $x^2 + y^2 = 1$ and evaluate

$$\iiint_V dV \quad (18)$$

Soln:



$$\text{top } z = 1 \Rightarrow \rho \cos \phi = 1$$

$$\text{side } x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1$$

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta + \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi d\rho d\phi d\theta$$