Hole in the Sphere

I first encountered this problem in a puzzle book by the famous mathematician Martin Gardner. His solution was interesting in that it avoided the use of the calculus. I wanted to present a solution that uses calculus, as it is more straightforward and reveals the solution perhaps more elegantly. The problem has a rather remarkable and unintuitive result. The problem was stated as follows:

> This is an incredible problem – incredible because it seems to lack sufficient data for a solution. A cylindrical hole six inches long has been drilled straight through the center of a solid sphere. What is the volume remaining in the sphere?

Indeed it seems at first glance to lack enough information about the sphere. But some thought about a cylindrical hole *through* a sphere reveals that once the hole is a certain depth, one finds oneself *outside* the sphere and the hole is complete. There is only a ring remaining with a cylindrical interior and a band of a partial sphere as its exterior.

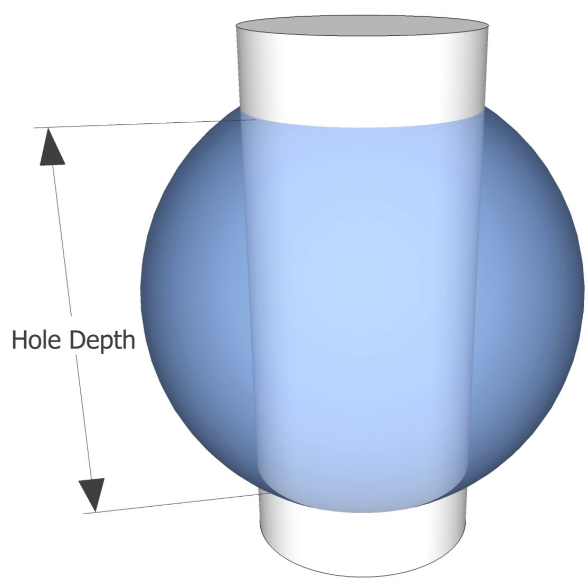

Figure 1. A cylindrical hole through a sphere.
Cylinder drawn for clarity.

In Fig. 1 I have drawn such a hole through a semi-transparent sphere with an extended cylinder representing the hole for clarity. The hole depth is marked and it can be clearly seen that the *hole diameter is fixed by the hole depth*. For example look at Figure 2, where a larger sphere has a hole of the same depth as in Fig. 1 but requires a larger diameter hole to accomplish it:

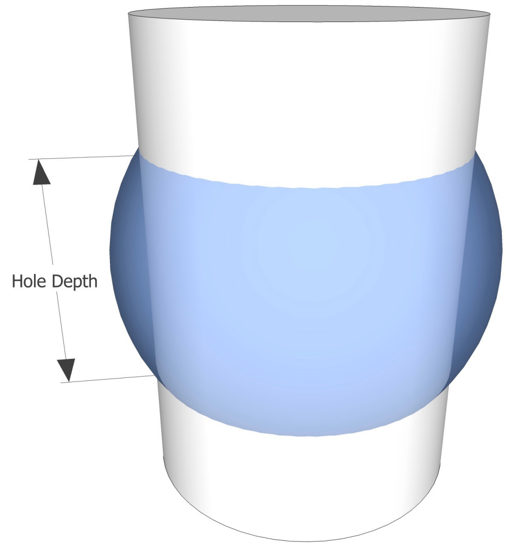

Figure 2. A larger sphere will require a larger diameter hole if the same hole depth as figure 1 is to be drilled through it.

This can be carried out for *any size* sphere, provided the diameter of the sphere is larger than the hole depth. For example a 6-inch hole could be drilled through a sphere the size of the Earth, leaving only an extremely thin, 6-inch wide band in the form of a ring with the diameter of the Earth.

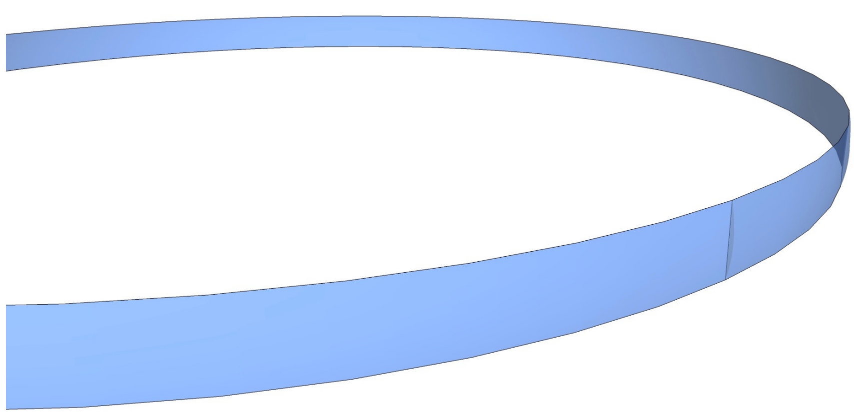

Figure 3. A thin band is all that remains of a large sphere with a shallow hole through it.

What is remarkable about such a volume residue from a sphere is that *it is not a function of the original sphere's radius!* This much can be gleaned from the way the problem was introduced. You are told there is enough information and yet the only way that could be true is if the original sphere's dimensions are not required to solve it. This fact will be proven later.

The Easy Solution

Since we may assume from the wording of the problem that the sphere's dimensions are not required, we may take the easy way out (as I first did) and consider the 6-inch cylindrical hole to be of infinitesimal diameter. That is, through a sphere of exactly 6 inch diameter. This way the hole is the right depth but effectively *removes no volume from the sphere*. That means the remaining volume is simply the volume of a six-inch diameter sphere:

$$V = \frac{4\pi}{3}3^3 = 36\pi$$

While this does give the answer, it is somewhat unsatisfying. For example we aren't shown why the size of the sphere is immaterial to the result. We are also not sure of a general equation for *any* depth hole.

The Better Solution

To see why a hole of a given depth through a sphere always leaves the same volume whether it's through a basketball or the Earth we will use calculus in cylindrical coordinates. This will also yield a general solution, as we will be leaving the hole depth as a variable.
The trickiest part of the problem is setting up the limits of the integrals. We will consider the depth of the hole to be l. Figure 4 shows a differential slice of the ring of a general volume residue.

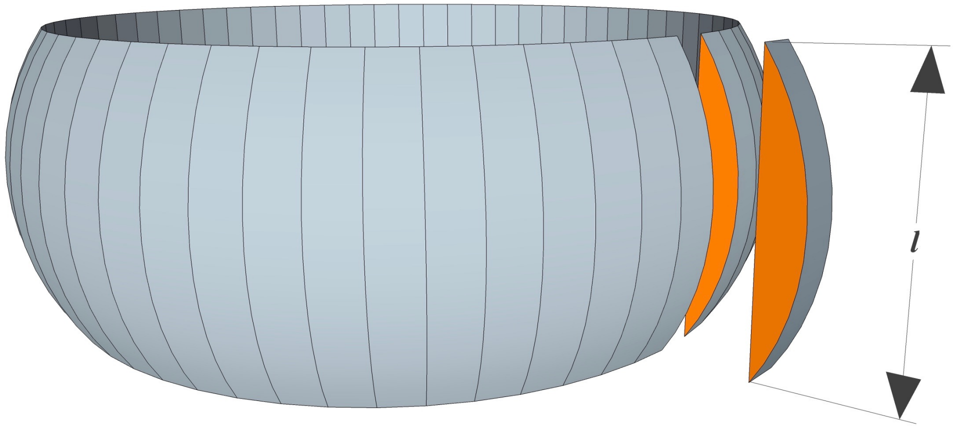

Figure 4. Differential slice of the remaining volume residue.

The interior cylindrical hole is parallel to the z-axis. The interior wall will be defined as the cylinder of radius a, but as we have seen when the hole depth is defined, a is fixed for a sphere of radius R. Let us find the limits of integration for the r-axis.

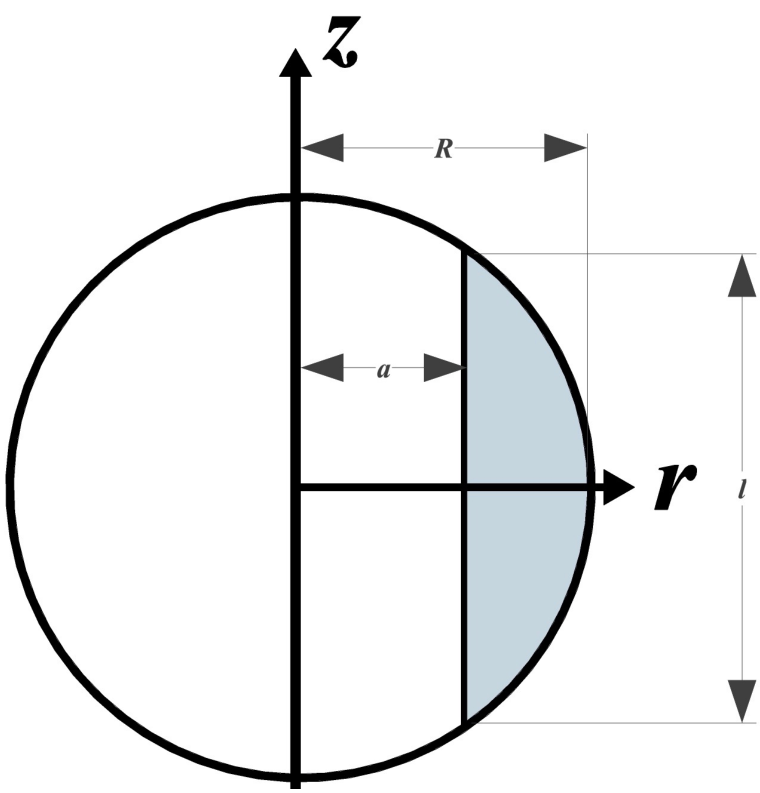

Figure 5. A cross section of the volume element (grey area). The cylindrical hole has radius *a,* and the sphere has radius *R.* Axes z and r are shown in bold.

It can be seen by using the well-known equation of a circle in the plane that our complete circle is given by

$$r^2 + z^2 = R^2$$

Solving for the variable *r*

$$r = \sqrt{R^2 - z^2} \qquad (1)$$

Equation 1 is the outer limit of integration for the variable r. It describes the circular projection of the sphere in cylindrical coordinates. It remains to find the inner limit for r, which is the wall of the cylindrical hole. Cylinders are trivially simple to describe in cylindrical coordinates, but we must find a way to write it in terms of the depth of the hole, l.
We can get to this best by considering another measurement on Fig. 5

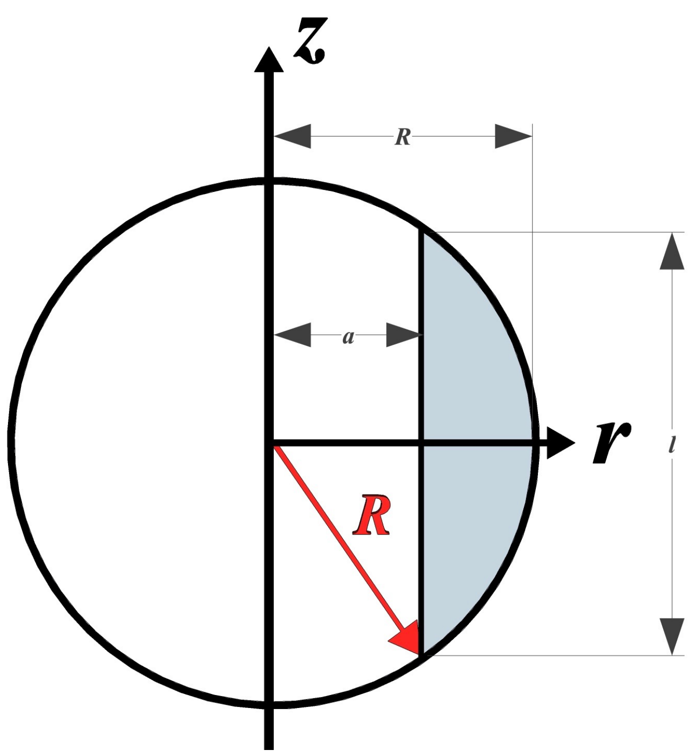

Figure 6. The red arrow cutting across the empty area inside the hole in the sphere is the radius *R.*

We can now see by the Pythagorean theorem that the spherical radius is also given by $R = \sqrt{a^2 + \left(\frac{l}{2}\right)^2}$. Since we need the value of a for our limit we solve for it instead

$$a = \sqrt{R^2 - \left(\frac{l}{2}\right)^2}$$

We now set up the volume integral as follows

$$V = \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{\sqrt{R^2-(l/2)^2}}^{\sqrt{R^2-z^2}} r\, drdzd\theta$$

Where the z-axis limits are symmetric about the point $z = 0$. Solving in steps gives

$$V = \int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{1}{2}[r^2]_{\sqrt{R^2-(l/2)^2}}^{\sqrt{R^2-z^2}} dzd\theta$$

$$V = \frac{1}{2}\int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[R^2 - z^2 - R^2 + \left(\frac{l}{2}\right)^2\right] dzd\theta$$

Note the sphere's radius R subtracts out

$$V = \frac{1}{2}\int_0^{2\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \left[\left(\frac{l}{2}\right)^2 - z^2\right] dzd\theta$$

$$V = \frac{1}{2}\int_{0}^{2\pi} \left[z\left(\frac{l}{2}\right)^2 - \frac{1}{3}z^3\right]_{-\frac{l}{2}}^{\frac{l}{2}} d\theta$$

$$V = \frac{1}{2}\int_{0}^{2\pi} \left\{\left(\frac{l}{2}\right)^3 - \frac{1}{3}\left(\frac{l}{2}\right)^3 - \left(-\left(\frac{l}{2}\right)^3 - \left(-\frac{1}{3}\left(\frac{l}{2}\right)^3\right)\right)\right\} d\theta$$

Cleaning up,

$$V = \frac{1}{2}\int_{0}^{2\pi} \left[2\left(\frac{l}{2}\right)^3 - \frac{2}{3}\left(\frac{l}{2}\right)^3\right] d\theta$$

$$V = \left(\frac{l}{2}\right)^3 \int_{0}^{2\pi} \left[1 - \frac{1}{3}\right] d\theta = \left(\frac{l}{2}\right)^3 \int_{0}^{2\pi} \frac{2}{3} d\theta$$

Carrying out the last integral gives the final expression for the volume of the remainder of the sphere:

$$\boxed{V = \frac{4\pi}{3}\left(\frac{l}{2}\right)^3}$$

This is indeed a remarkable result, as it is independent of the size of the sphere involved. *The remaining volume of a sphere with a hole of depth l drilled through it is equal to a sphere with diameter l.* Looking at the original problem statement of a 6-inch hole we see the result is again 36π, as seen in the simpler solution.