# Math 1497 - Calc 2

## Infinite Series

$$\sum_{n=1}^{\infty} a_n \Leftarrow \text{does it this converge?}$$

Before we create a battery of tests to ~~deter~~ determine this we consider some special series

(1) **Geometric**

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

converges if $|r| < 1$ if so $S_\infty = \frac{a}{1-r}$

(3) **Harmonic**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{diverges}$$

#2 **Telescopic**

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Stop at $N$

$$S_N = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

drop ( ) & expand

$$= 1 - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} \cdots \cancel{\tfrac{1}{N-1}} - \cancel{\tfrac{1}{N}} + \cancel{\tfrac{1}{N}} - \frac{1}{N+1}$$

$$= 1 - \frac{1}{N+1}$$

Now let $N \to \infty$ $\quad \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 - \frac{1}{N+1} = 1$

so this series conv.

In general, for telescopic series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n - b_{n+1} \qquad a_n \text{ splits}$$

(1) Stop at $N$

$$S_N = (b_1 - b_2) + (b_2 - b_3) + \cdots (b_N - b_{N+1})$$

(2) drop ( ) & expand

(3) cancel terms

(4) $N \to \infty$

ex $\sum_{n=1}^{\infty} \ln \frac{n}{n+1} = \sum_{n=1}^{\infty} \ln n - \ln n+1$

(1) Stop at $N$

$S_N = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots (\ln N-1 - \ln N)$
$+ (\ln N - \ln N+1)$

$= \ln 1 - \ln N+1$

$\lim_{N\to\infty} S_N = - \lim_{N\to\infty} \ln N+1 \to -\infty$ so div

After you cancel terms, end part of what's left will determine convergence / divergence

<u>Test # 4</u> p series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $p > 1$
diverges $p \leq 1$

Now we are ready to create some tests to determine convergence & divergence

<u>Test 1</u> $n^{th}$ term test

if $\lim_{n\to\infty} a_n = \#$ not zero the series diverges

Note: If a series converges

$$\lim_{n\to\infty} a_n = 0$$

to show this consider

$$S_{n-1} = a_1 + a_2 + \cdots a_{n-1}$$

$$S_n = a_1 + a_2 + \cdots a_{n-1} + a_n$$

$$= S_{n-1} + a_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{N-1} + \lim_{n\to\infty} a_n$$

$$L = L + \lim_{n\to\infty} a_n \Rightarrow \lim_{n\to\infty} a_n = 0$$

ex 1

$$\sum_{n=1}^{\infty} \frac{n}{n+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

$> \frac{1}{2}$ $\quad > \frac{1}{2}$ $\quad > 1/2$

Adding up an infinite # 1/2's

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

so by the nth term test the series div

Ex 2 $\sum_{n=2}^{\infty} \frac{n}{\ln n}$ $\lim_{n\to\infty} \frac{n}{\ln n} = \text{"}\frac{\infty}{\infty}\text{"}$

LH $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \lim_{x\to\infty} x \to \infty \neq 0$

so series div

Ex 3 $\sum_{n=0}^{\infty} \frac{1-2^n}{1+2^n}$ (UT - Knoxville)

$\lim_{x\to\infty} \frac{1-2^x}{1+2^x} = \lim_{x\to\infty} \frac{\frac{1}{2^x} - 1}{\frac{1}{2^x} + 1} = -1 \neq 0$ series div by nth term test

OR LH $\lim_{x\to\infty} \frac{-2^x \ln 2}{2^x \ln 2} = -1$ same conclusion

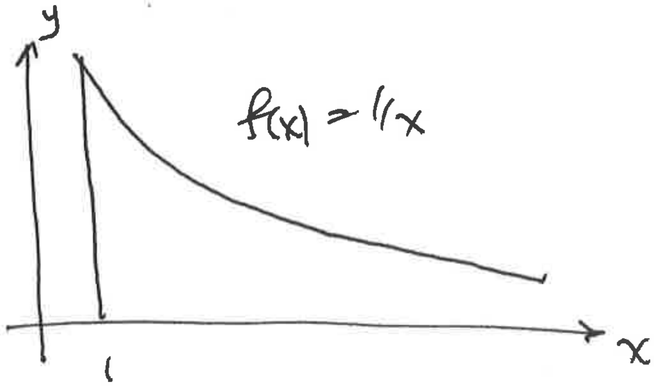
Ex 4 $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

what can we say – nothing!

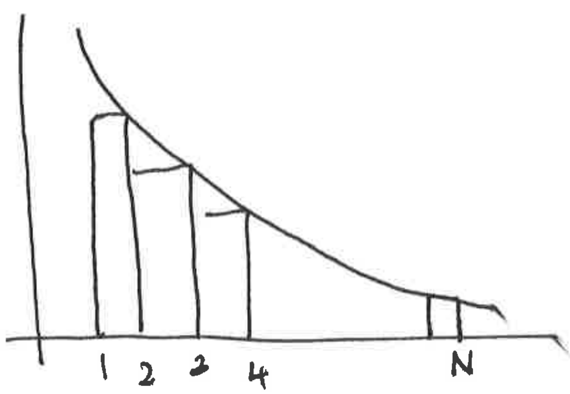
This test doesn't determine conv. – more later!

## Test 2 Integral Test

Motivation - consider $f(x) = \frac{1}{x}$



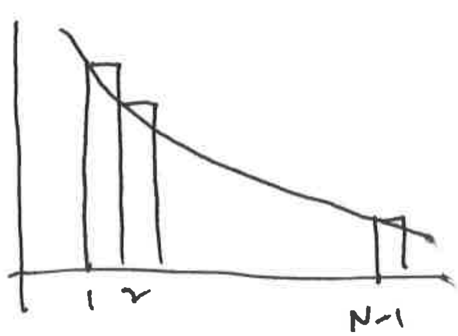

Area's under - step size 1




$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{N} < \int_1^N \frac{dx}{x}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{N} < 1 + \int_1^N \frac{dx}{x}$$

Area over




$$\int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{N-1}$$

$$\frac{1}{N} + \int_1^N \frac{dx}{x} < 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{N}$$

so $\frac{1}{N} + \int_1^N \frac{dx}{x} < \sum_{n=1}^{N} \frac{1}{n} < 1 + \int_1^N \frac{dx}{x}$

Now $N \to \infty$

$$\int_1^\infty \frac{dx}{x} < \sum_{n=1}^{\infty} \frac{1}{n} < 1 + \int_1^\infty \frac{dx}{x}$$

$\therefore$ $\int_1^\infty \frac{dx}{x}$ diverges – improper $\int$

the series diverges

This is the integral test

<u>Test 2</u> $\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$ $\int_1^\infty f(x)\,dx$

If $f$ is positive, decreasing cont^s

test applies

if $\int_1^\infty f(x)\,dx$ conv (div) $\sum_{n=1}^{\infty} a_n$ conv/div.

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ex $\sum_{n=2}^{\infty} \frac{n}{n^2+1}$

let $f(x) = \frac{x}{x^2+1}, \quad x \geq 2$ Yes cnt$^s$ ✓

Yes $> 0$

$$f' = \frac{1(x^2+1) - 2x \cdot x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \text{ for } x \geq 2$$

So test applies

$$\int_2^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_2^b \frac{x}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln|x^2+1| \Big|_2^b$$

$$= \lim_{b\to\infty} \frac{1}{2} \ln|b^2+1| - \frac{1}{2}\ln 5 \to \infty$$

$\therefore$ $\int$ diverges by $\int$ test series div

Ex $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$f(x) = \frac{1}{x(\ln x)^2}$

(1) $f$ cont$^s$ ✓

(2) $f > 0$ ✓

(3) $f' = \frac{0 \cdot x(\ln x)^2 - 1\left[(\ln x)^2 + x \cdot 2\ln x \cdot \frac{1}{x}\right]}{(x \ln^2 x)^4}$

$< 0$ ✓

so test applies

$$\int_2^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{b\to\infty} \int_2^b \frac{dx}{x(\ln x)^2}$$

sub $u = \ln x$

$$= \lim_{b\to\infty} -\frac{1}{\ln x}\Big|_2^b = \lim_{b\to\infty} -\frac{1}{\ln b} + \frac{1}{\ln 2} \to \frac{1}{\ln 2}$$

so by integral test series converge

not the actual sum

Final Note:

$\lim_{n\to\infty} a_n = 0$ means nothing

$\sum_{n=1}^{\infty} \frac{1}{n}$ div. $\quad \lim_{n\to\infty} \frac{1}{n} = 0$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv $p=2$ $\quad \lim_{n\to\infty} \frac{1}{n^2} = 0$