VANDERBILT UNIVERSITY School of Engineering

# Discrete Structures

## CS 2212

(Fall 2020)

## 10 – Functions

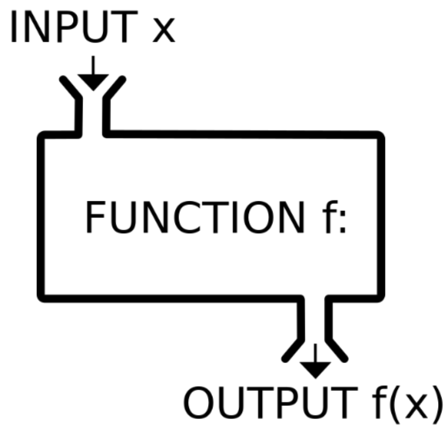
# Chapter 4

# Functions

# Functions

In computer programming a **function** is a method that returns a single result.

**Example:** A function to compute the square of a number. Accepts a value, squares it and returns the square.



# Functions and Terminology

In discrete structures, a **function *f*** from L to M associates each element of L with exactly one element of M.

$$f : L \to M$$

We call **L** the **domain**

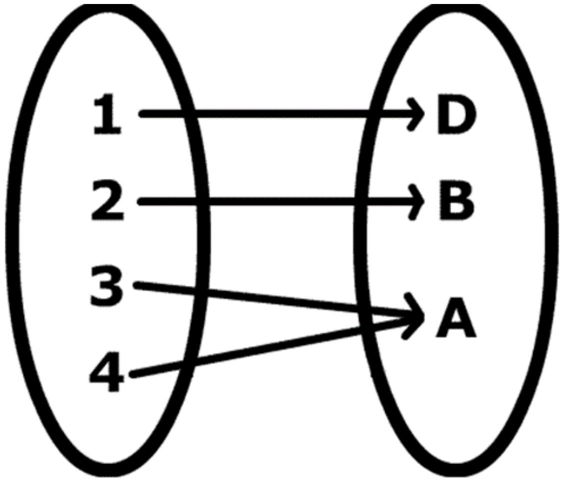
We call **M** the **codomain**

**Translation:**

- Everyone from the domain maps to someone in the codomain.
- No one from domain maps to more than one thing in the codomain.

# Functions

Does the following mapping from set L to set M represent a valid function?




**Yes.**

- Everyone maps to someone.
- No one maps to more than one thing.

# Functions: Floor and Ceiling

The floor and ceiling functions are of type **R** → **Z**

**floor(x)** is the closest integer less than or equal to x.
It is notated as $\lfloor x \rfloor$ or floor(x)

**ceiling(x)** is the closest integer greater than or equal to x. It is notated as $\lceil x \rceil$ or ceiling(x)

# Functions: Floor and Ceiling

**Floor:**

$\lfloor 2.6 \rfloor = 2$

$\lfloor -2.1 \rfloor = -3$ (visualize the number line)

**Ceiling:**

$\lceil 2.6 \rceil = 3$

$\lceil -2.1 \rceil = -2$ (visualize the number line)

# Floor and Ceiling Properties

**Property:** $\lceil x + 1 \rceil = \lceil x \rceil + 1; \quad x \in \mathbf{R}$

**Proof:**

| | | |
|---|---|---|
| 1. | Assume $x \in$ R | premise |
| 2. | There exists an integer n; $n < x \leq n + 1$ | definition of **R, Z** |
| 3. | So $n + 1 < x + 1 \leq n + 2$ | adding 1 |
| 4. | So $\lceil x + 1 \rceil = n + 2$ | ceiling definition |
| 5. | $\lceil x + 1 \rceil = n + 1 + 1$ | simplifying |
| 6. | Also $\lceil x \rceil = n + 1$ | from line 2 |
| 7. | $\lceil x + 1 \rceil = \lceil x \rceil + 1$ | from lines 5, 6 |
| 8. | QED | 1 - 7 |

# Floor and Ceiling Properties

**Question:** Is the ceil ($x$) = floor($x$) + 1; $x \in Z$?

**Answer:**

No, disprove by counterexample that the ceil ($x$) does not equal floor($x$) + 1 for $x \in Z$.

1. Let $x \in Z$ be the number 5
2. ceil (5) = 5
3. floor(5) + 1 = 5+1 = 6
4. 5 is not equal to 6
5. ceil ($x$) does not equal floor($x$) + 1

# Functions and Terminology

$\boldsymbol{f(x) = y}$ means

$f$ associates $x \in$ L (domain) with $y \in$ M (co-domain).

We say: $f$ of $x$ is $y$ or $f$ maps $x$ to $y$.

$f$ and $g$ are **equal** functions if $f(x) = g(x)$ for all $x \in$ L

The **range( $f$ )** is the set of values in the co-domain that are explicitly mapped from the domain.

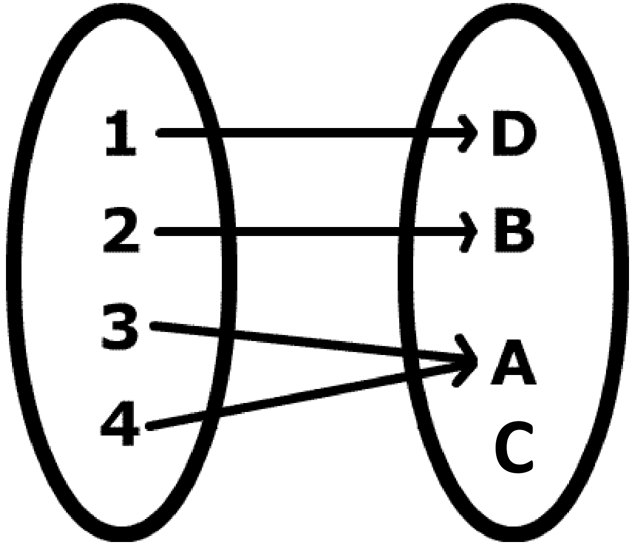
The range is a subset of the codomain.

# Functions and Terminology

**Question:** The picture below shows a function $f : L \to M$. What is the domain, codomain and range?

**Answer:**

- **Domain:** $L = \{1, 2, 3, 4\}$
- **Codomain:** $M = \{D, B, A, C\}$
- **range( $f$ ):** $\{A, B, D\}$



# Functions and Terminology

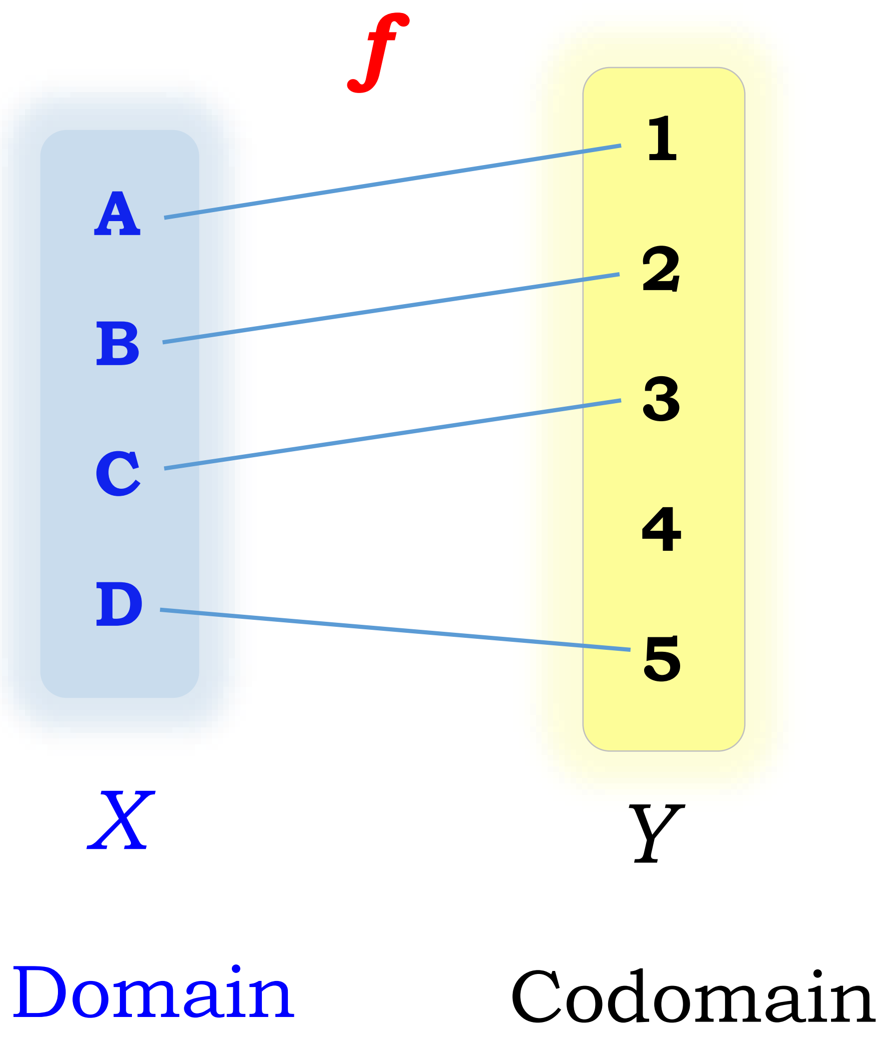
Evaluating a function at each element of a **subset X** of the domain, produces a set called the **image** of X

The **image** of X is denoted by: $f(X) = \{f(x) \mid x \in X\}$

The **pre-image** (i.e., inverse image) of a subset Y of the codomain is the set of all elements of the domain that map to the members of Y

The pre-image is denoted by : $f^{-1}(Y) = \{x \mid f(x) \in Y\}$.

# Recap: Functions and Terminology




**Range** $(f) = \{1, 2, 3, 5\}$

**Image** of $\{C, D\} = f(\{C, D\})$

$= \{3, 5\}$

**Pre-image** of $\{4, 5\} = f^{-1}(\{4, 5\})$

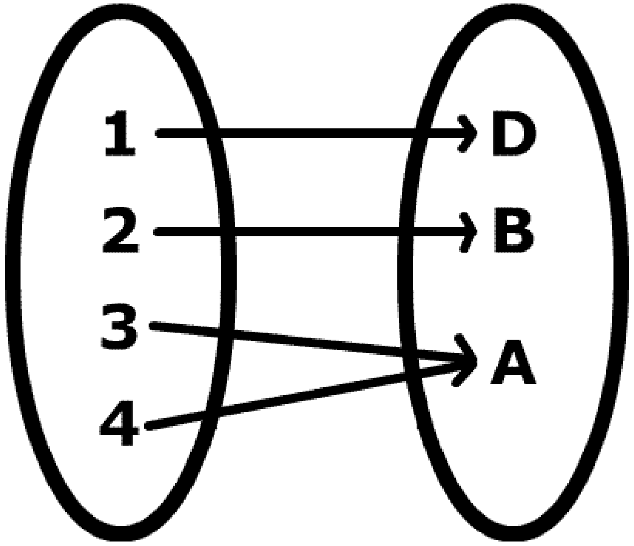
$= \{D\}$

# Injective Functions

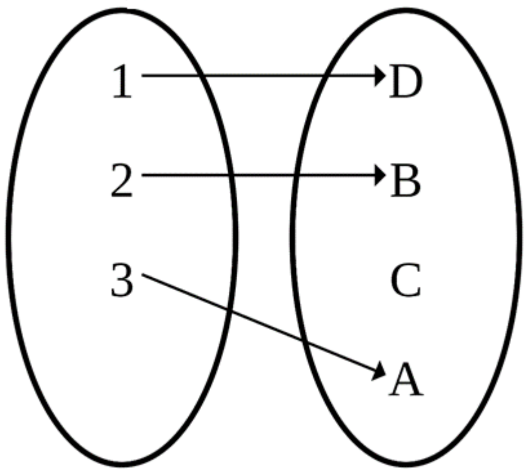
**Functions that have special properties** are useful in solving a variety of problems.

A function is injective or **one-to-one** if distinct elements in A map to distinct elements in B. In other words, no two items map to the same element.

**Not an injection/one-to-one**




**Injection/one-to-one**

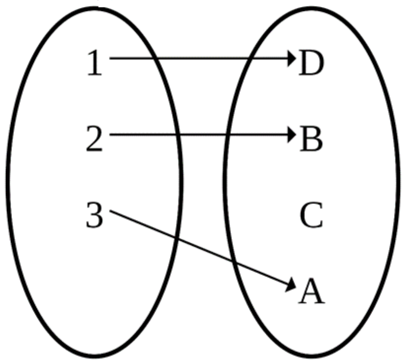


# Injective Functions

**Injective Functions:**

Let $f : A \to B$ be a function.

We say that $f$ is **Injective (also called one-to-one)**, if $x \neq y$ implies $f(x) \neq f(y)$ or, equivalently, $f(x) = f(y)$ implies $x = y$



# Injective Functions

**Question:**

Let ***f*** be the function that assigns letter grades (A, B, C, D, F) at Vanderbilt to a group of 35 students in CS 2212. Is ***f*** injective?

**Answer:**

No. At least one grade will have to be duplicated because there are 35 students but not 35 letter grades.

# Injective Functions

**Question:**

Let $f : \mathbf{Z} \rightarrow \mathbf{N}$ be defined by $f(x) = x^2$. Is $f$ injective?

**Answer:**

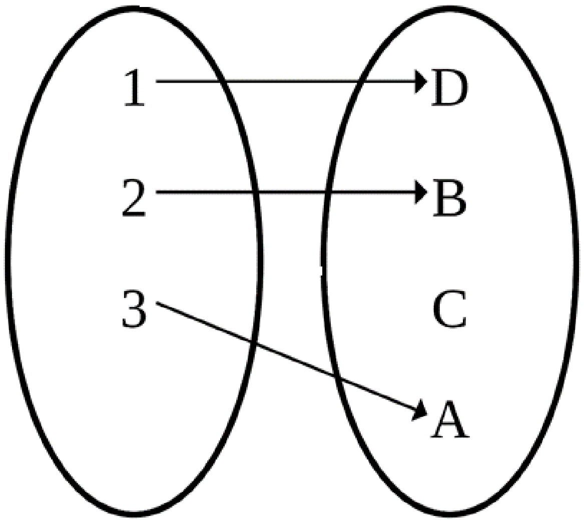
No.

For example $f(2) = f(-2)$. So the mapping from $Z \rightarrow N$ is not unique for f. Therefore $f$ is not injective.

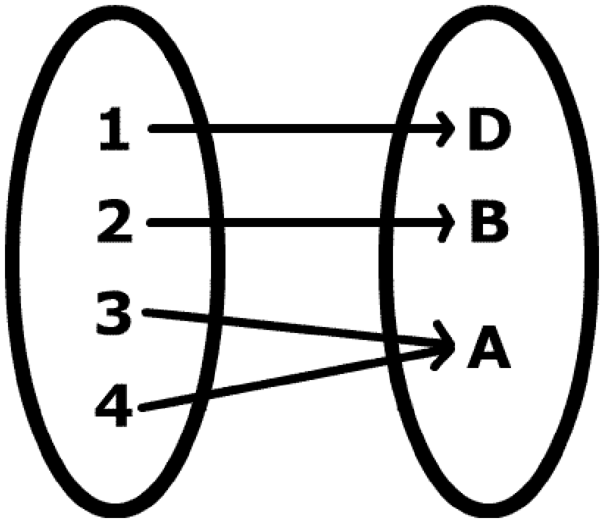
# Surjective Functions

A function is **surjective** or **onto** if every element in the codomain is mapped by ***f***. This mapping does **not** have to be unique (but it can be).

**Not surjective/not onto**




**Surjective/onto**

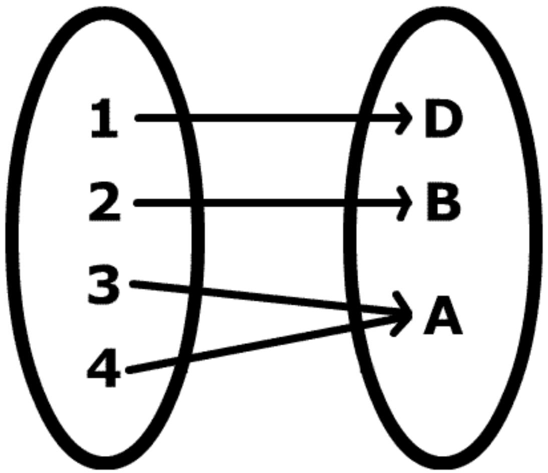


# Surjective Functions

**Surjective Functions:**

Let $f : A \to B$ be a function.

We say that f is **Surjective (also called onto)**, if each b ∈ B has the form b = $f$(a) for some a ∈ A.



# Surjective Functions

**Question:**

There are 45000 people waiting in line for 35000 seats for football match.

Let ***f*** be the function that maps people to seats available at the match. Is ***f*** surjective?

**Answer:**

Yes. Since there are more people than seats, we know that all seats will be filled. Therefore ***f*** is surjective.

# Surjective Functions

**Question:**

There are 10000 people waiting in line for 35000 seats for football match.

Let $f$ be the function that maps people to seats available at the match. Is $f$ surjective?

**Answer:**

No. Since there are more seats than people to fill the seats, some seats will be empty. Therefore $f$ is not surjective.

# Surjective Functions

**Question:**

$f : \mathbf{Z} \rightarrow \mathbf{N}$ defined by $f(x) = x^2$. Is $f$ surjective?

**Answer:**

No. $f$ is not surjective. There is no mapping for element 3 in N from $f$ since 3 is not a square.

# Question

For each property below, define a **function** that satisfies the property. You can choose the domain and codomain for each function from the following two sets:

$$\mathbf{A} = \{\mathbf{a, b, c, d}\}$$

$$\mathbf{B} = \{\mathbf{x, y}\}$$

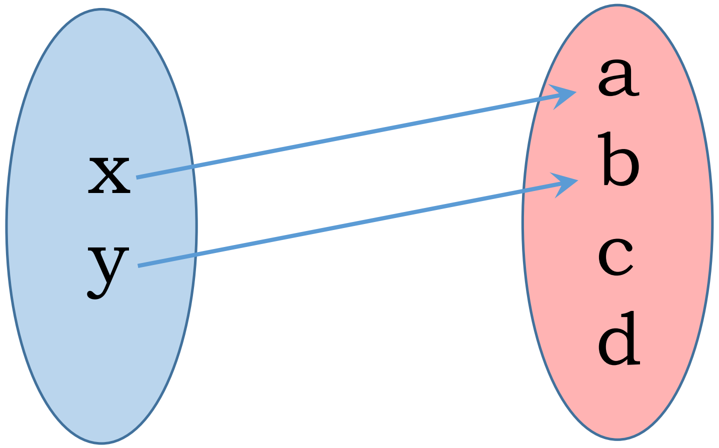
and you can choose to map B → A or A → B.

**Injective but not surjective**

How about $f$: B → A defined as

$f\,(\mathrm{x}) = \mathrm{a}$;

$f\,(\mathrm{y}) = \mathrm{b}$



# Question

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$$\mathbf{A} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$$

$$\mathbf{B} = \{\mathbf{x}, \mathbf{y}\}$$

and you can choose to map $B \rightarrow A$ or $A \rightarrow B$.

**Surjective but not injective**

# Question

For each property below, define a **function** that satisfies the property. You can choose the domain and codomain for each function from the following two sets:

$$\mathbf{A} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$$

$$\mathbf{B} = \{\mathbf{x}, \mathbf{y}\}$$

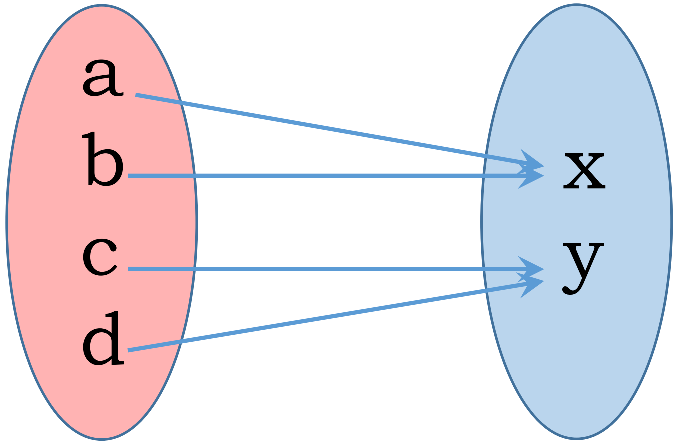
and you can choose to map B → A or A → B.

**Surjective but not injective**

How about $f$: A → B defined as

$f(a) = f(b) = x$ ;

$f(c) = f(d) = y$;



# Question

For each property below, define a **function** that satisfies the property. You can choose the domain and codomain for each function from the following two sets:

$$A = \{a, b, c, d\}$$

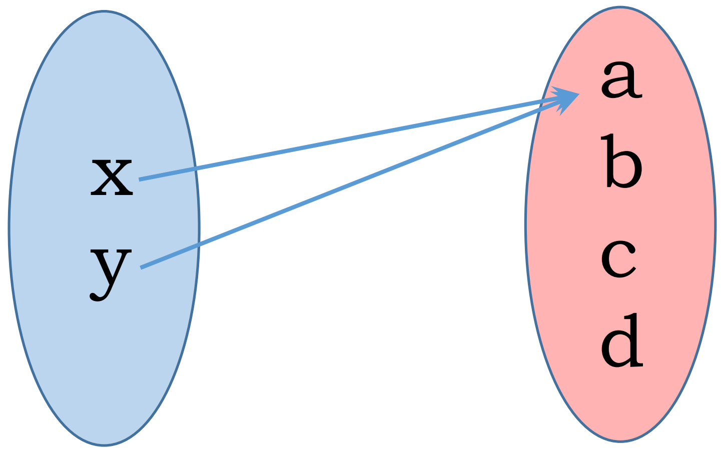
$$B = \{x, y\}$$

and you can choose to map $B \to A$ or $A \to B$.

**Neither surjective nor injective**

How about $f: B \to A$ defined as

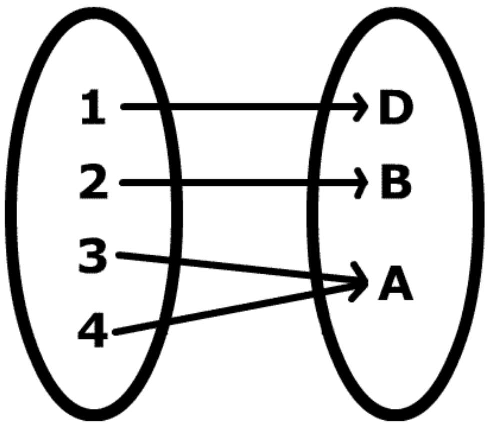
$f(x) = f(y) = a$ ;



# Bijective Functions

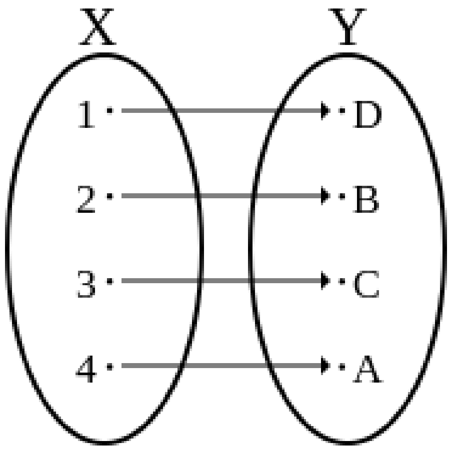
A function is bijective if there is a perfect "one-to-one” correspondence between the members of the domain and codomain.

**Not bijective**

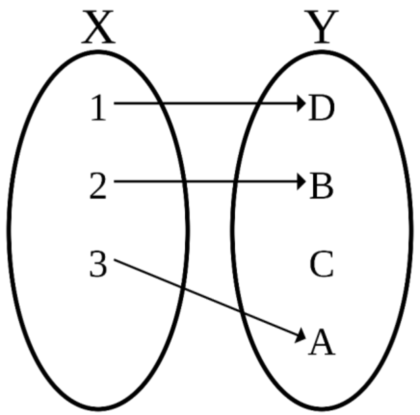



**(Surjective but not injective)**

**Bijective**




**Not bijective**



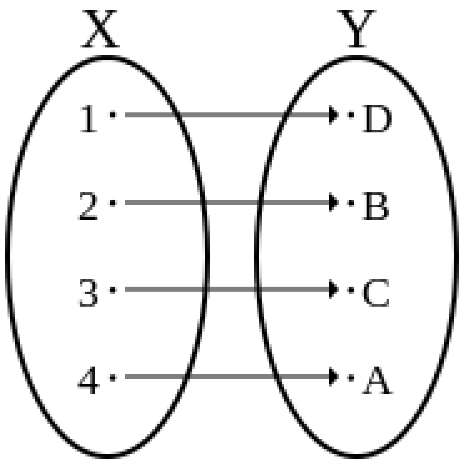

**(Injective but not surjective)**

# Bijective Functions

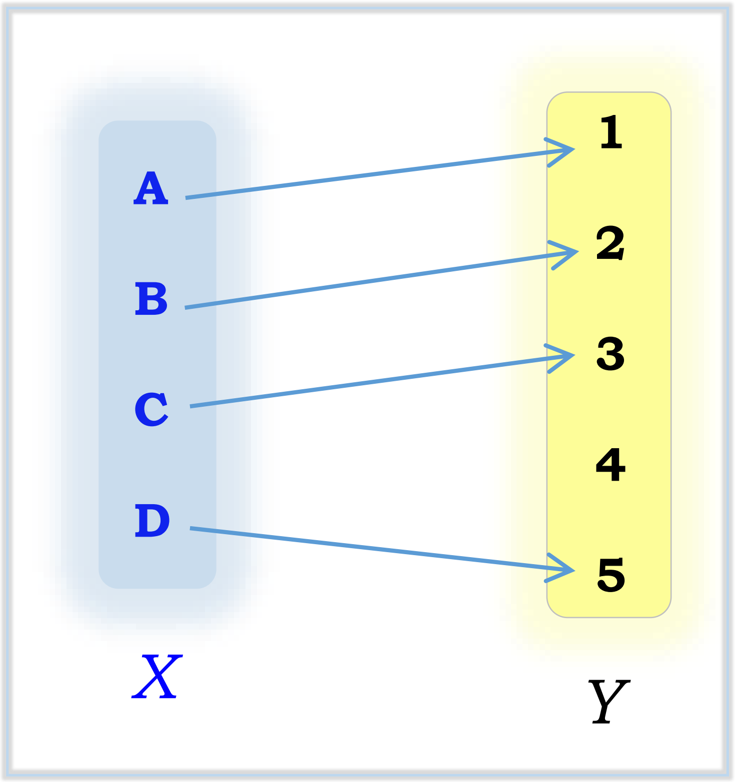
**Bijective Functions:**

Let $f : A \to B$ be a function.

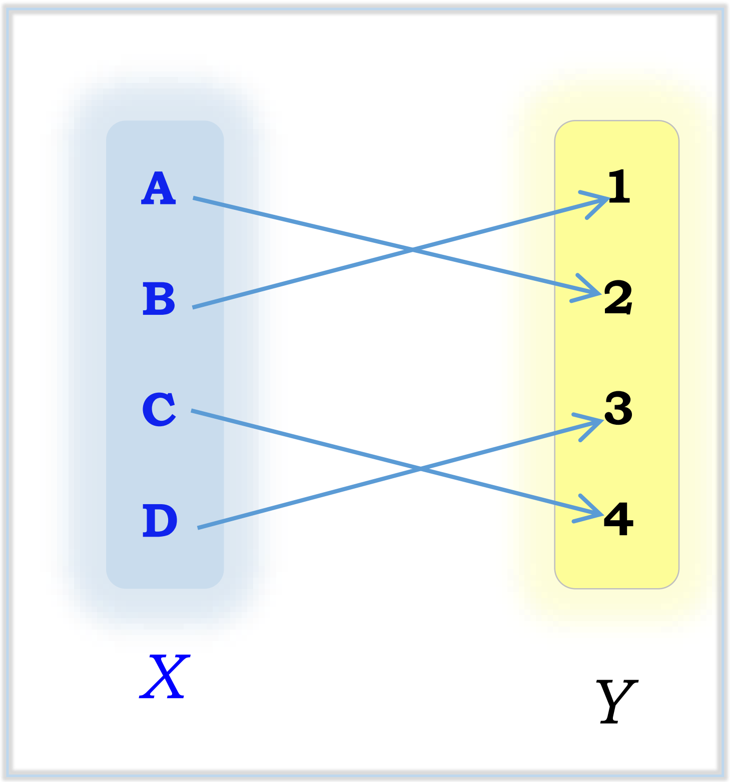
We say that $f$ is **Bijective (also called one-to one and onto)**, if $f$ is both injective and surjective at the same time.



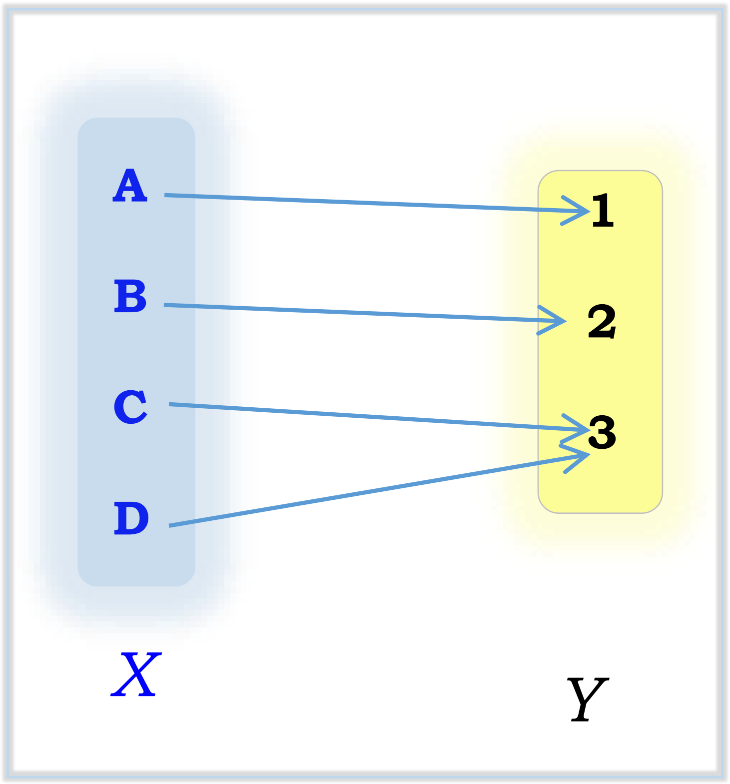
# Recap: Functions




Injective
(|$X$| ≤ |$Y$|)




Bijective
(|$X$| ≤ |$Y$|)




Surjective
(|$X$| ≥ |$Y$|)

# How to Show a Function is Injective?

To prove: $f : A \to B$ is injective

Key Point:

Let $x \in A$ and $f(x)$ is an image of $x$.

$y \in A$ and $f(y)$ is an image of $y$.

For injectivity of $f$, all we need to show is:

If $f(x) = f(y)$, then $x = y$.

# How to Show a Function is Injective?

Let X: set of non-zero real numbers

Show that: $f : X \to X$ is injective,

where, $f(x) = 1/x$.

Let $x, y \in X$.

Then, $f(x) = 1/x$ and $f(y) = 1/y$.

(We need to show that $f(x) = f(y)$ implies $x = y$.)

$f(x) = f(y)$ implies that $1/x = 1/y$,

which further implies that $x = y$.

Hence, $f$ is injective.

# How to Show a Function is Surjective?

To prove: $f : A \to B$ is surjective

Key Point: Fix any $y \in B$

We need to show: there must exist some $x \in A$ whose image is $y$,
(i.e., there exists $x \in A$ such that $f(x) = y$.)

Then proceed as follows:

- Consider $x =$ some expression in terms of $y$ (express $x$ in terms of $y$)
- Show that $x \in A$.
- Then show that $f(x) = y$.

# How to Show a Function is Injective?

Let X: set of non-zero real numbers

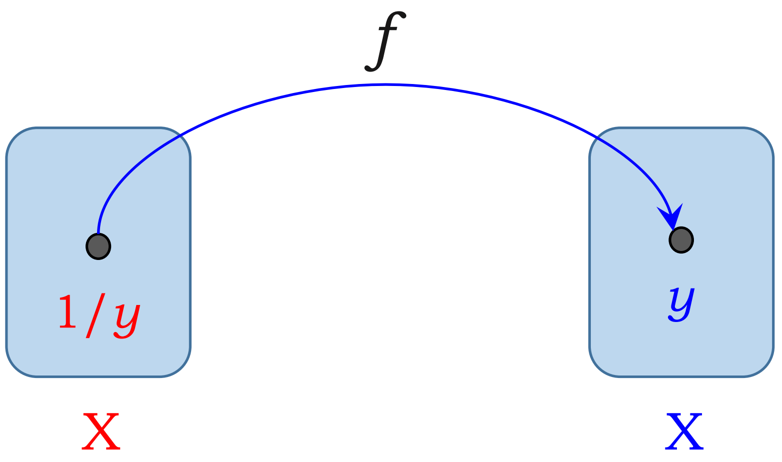
Show that: $f : X \to X$ is surjective,

where, $f(x) = 1/x$.

Let $y \in X$ (co-domain).

We need to show that $\exists x$ in the domain such that $f(x) = y$.

| | | |
|---|---|---|
| 1. | Let $x = 1/y$ | (express $x$ in terms of $y$) |
| 2. | $y \in X$ means that $(x = 1/y) \in X$ | (show that $x$ is in the domain). |
| 3. | $f(x) = f(1/y) = y$ | (show that $f(x) = y$) |



# Another Example: Surjective Function

Let X: set of non-negative real numbers

Show that: $f : X \to X$ is surjective,

where, $f(x) = x^2$.

**Goal:** We need to show that for every $y \in X$ (in the co-domain) there exists an $x$ (in the domain), such that

$$f(x) = y.$$

Let $y \in X$ (co-domain, i.e. $y$ is a non-negative real number)

# Another Example: Surjective Function

We need an $x$ (element in the domain) to start with. How to get that $x$ ?

From $y = f(x)$, obtain an expression for $x$.

Step 1: $y = f(x) = x^2$.

From here, we get $x = \pm\sqrt{y}$ . So, we pick $x = \sqrt{y}$

Now show that $x$ (that we have obtained) is indeed in the domain

Step 2: If $y$ is a non-negative real number, then $x = \sqrt{y}$ is also a non-negative real number. Hence $x$ is in the domain

Now finally show that for $x$ (that we have obtained): $f(x) = y$

Step 3: $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$.