

Calculus 3 - Limits

In Calculus 1 we considered limits. For example

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 2}{2x + 1}.$$

If we directly substitute $x = 1$ we get

$$\lim_{x \rightarrow 1} \frac{x^2 + 2x - 2}{2x + 1} = \frac{1^2 + 2 \cdot 1 - 2}{2 \cdot 1 + 1} = \frac{1}{3}$$

a simple number. However, if we consider

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0} \quad (1)$$

with $\frac{0}{0}$ meaning nothing. So we consider alternate approaches to derive the limit. For example, graphically we see that (1) and from the graph we

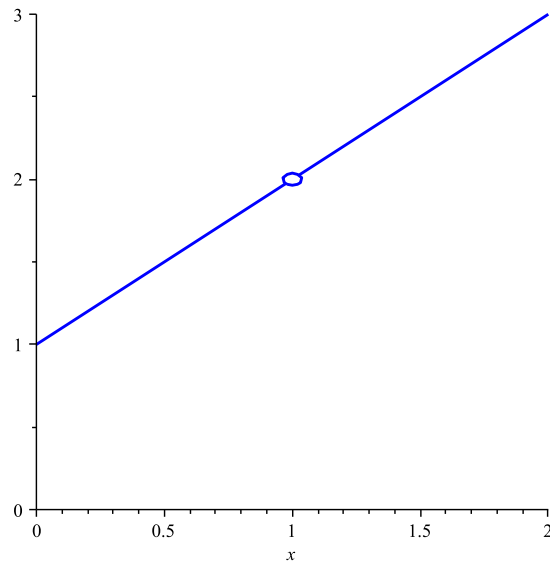


Figure 1: Graph of $f(x) = \frac{x^2 - 1}{x - 1}$

determined that

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2. \quad (2)$$

Analytically, we see

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} x + 1 = 2 \quad (3)$$

It's important to realize that not all limits exist. For example consider

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \quad (4)$$

and the limit

$$\lim_{x \rightarrow 0} f(x) \quad (5)$$

Clearly approaching zero from the left and from the right gives different

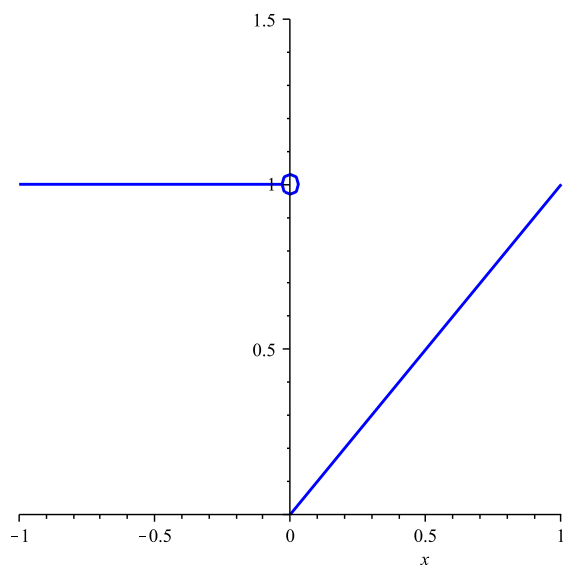


Figure 2: Branch function (4)

values. So we created one-sided limits

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0} 1 = 1 \\ \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0} x = 0\end{aligned}\tag{6}$$

and since

$$\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)\tag{7}$$

then the limit does not exist (DNE).

In general we considered

$$\lim_{x \rightarrow a} f(x) = L\tag{8}$$

and a very formal way (using $\delta - \epsilon$) of proving that limits exist.

So now we extend limits to 3D and consider

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y)\tag{9}$$

and ask - do these limits exist?

Consider for example,

$$\lim_{(x,y) \rightarrow (1,-1)} \frac{10xy - 2y^2}{x^2 + y^2}\tag{10}$$

Well, as a first approach, let's try a direct substitution. Doing so yields

$$\lim_{(x,y) \rightarrow (1,-1)} \frac{10xy - 2y^2}{x^2 + y^2} = \frac{-10 - 2}{1 + 1} = \frac{-12}{2} = -6\tag{11}$$

and so the limit is equal to 6.

Consider

$$\lim_{(x,y) \rightarrow (1,-2)} \frac{y^2 + 2xy}{y + 2x} \quad (12)$$

a direct substitution yields

$$\lim_{(x,y) \rightarrow (1,-2)} \frac{y^2 + 2xy}{y + 2x} = \frac{4 - 4}{-2 + 2} = \frac{0}{0} \quad (13)$$

so there's that $\frac{0}{0}$ again meaning we need to do something else. One will notice that factoring will work here since

$$\begin{aligned} \lim_{(x,y) \rightarrow (1,-2)} \frac{y^2 + 2xy}{y + 2x} &= \lim_{(x,y) \rightarrow (1,-2)} \frac{y(y + 2x)}{y + 2x} \\ &= \lim_{(x,y) \rightarrow (1,-2)} y \\ &= -2 \end{aligned} \quad (14)$$

so the limit exists!

Consider

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \frac{0}{0} \quad (15)$$

so we need to do something else. How many ways can we approach $(0,0)$? Well, there really are an infinite number of ways. For example, we could let $y = 0$ and then let $x \rightarrow 0$ so we would be approaching $(0,0)$ along the x-axis. We could let $x = 0$ and then let $y \rightarrow 0$ so we would be approaching $(0,0)$ along the y-axis. We could also let $y = x$ and then let $x \rightarrow 0$ so we would be approaching $(0,0)$ along the line $y = x$.

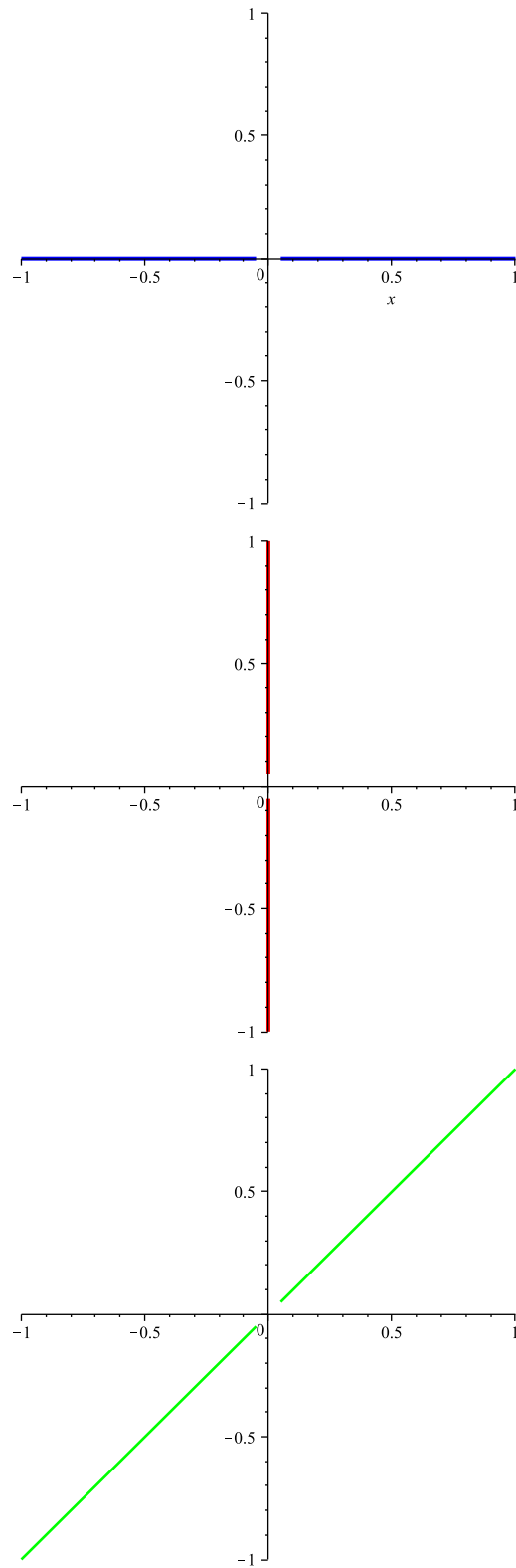


Figure 3: Following different paths

So let's see what happens in our limit ((23)) following the x and y axes.

$$\begin{aligned} \text{along } x \text{ axis } (y = 0) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - 0^2}{x^2 + 0^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{x^2} = 1 \\ \text{along } y \text{ axis } (x = 0) \quad \lim_{(x,y) \rightarrow (0,0)} \frac{0^2 - y^2}{0^2 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{-y^2}{y^2} = -1 \end{aligned} \quad (16)$$

and since following different paths, we get different limits, the limit DNE!

Example 4

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \frac{"0"}{"0"} \quad (17)$$

Along $x = 0$ (y axis) and $y = 0$ (x axis) we obtain

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{0}{y^2} = 0 \\ \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{0}{x^2} = 0 \end{aligned} \quad (18)$$

and so you might be tempted to say the limit is 0 but if we follow $y = x$ then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0 \quad (19)$$

and so in this example, the limit DNE!

Example 5

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} = \frac{"0"}{"0"} \quad (20)$$

Along $x = 0$, $y = 0$ and $y = x$ we get

$$\begin{aligned}\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} &= 0 \\ \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} &= 0 \\ \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} &= \lim_{(x,y) \rightarrow (0,0)} \frac{2x^3}{x^4 + x^2} = 0\end{aligned}\tag{21}$$

and so you might be tempted to say the limit is 0 but if we follow $y = x^2$ then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2} = \lim_{(x,y) \rightarrow (0,0)} \frac{2x^4}{x^4 + x^4} = \frac{2}{2} = 1 \neq 0\tag{22}$$

and so in this example, the limit DNE!

Example 6

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y^2}{x^2 + y^2} = \frac{"0"}{"0"}\tag{23}$$

Along $x = 0$, $y = 0$ and $y = x$ we get the limit is zero. So maybe the limit is actually zero. This we consider in the next class.