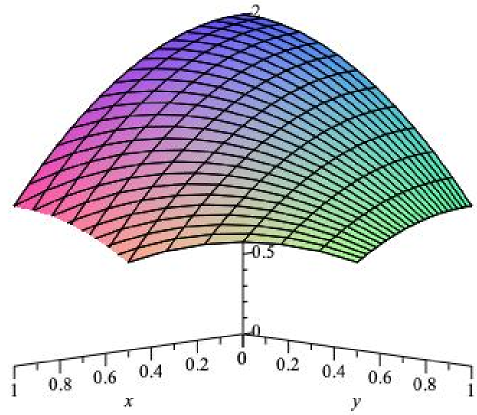
Calculus 3 - Volumes 2

We now continue our discussion using double integrals to find the volume under a surface where



$$V = \iint_R f(x,y)\,dA \tag{1}$$

where $dA = dxdy$ or $dydx$.

Example 1. Find the volume under the plane $2x + y + z = 2$ for $x, y, z \geq 0$. The volume is shown in fig. 1 in the first octant.

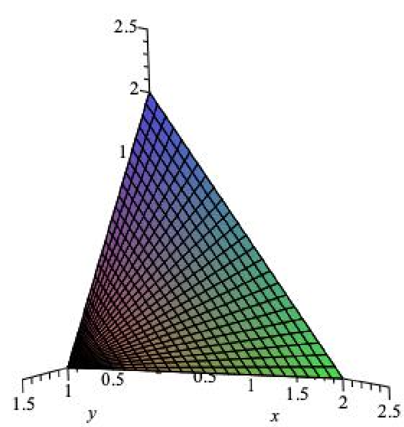
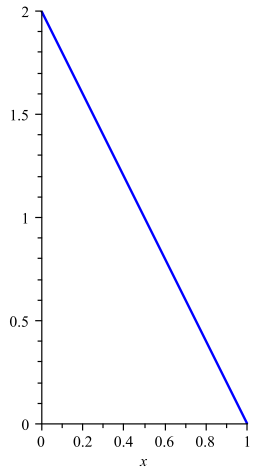


Figure 1: Surface



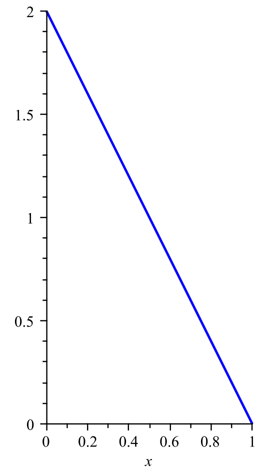


Figure 2: Region of integration

so the volume is

$$V = \int_0^1 \int_0^{2-2x} (2-2x-y)dydx \tag{2}$$

or

$$V = \int_0^2 \int_0^{\frac{2-y}{2}} (2-2x-y)dxdy \tag{3}$$

and we integrate so

$$\begin{aligned} V &= \int_0^1 \int_0^{2-2x} (2-2x-y)dydx \\ &= \int_0^1 \left(2y - 2xy - \frac{1}{2}y^2\right) \Big|_0^{2-2x} dx \\ &= \int_0^1 2(2-2x) - 2x(2-2x) - \frac{1}{2}(2-2x)^2\, dx \\ &= \int_0^1 2 - 4x - 2x^2 dx \\ &= 2x - 2x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{2}{3}. \end{aligned} \tag{4}$$

Example 2. pg 987, #26 Find the volume under the parabolic cylinder $z = 4 - y^2$ on the region bound by $y = x$, $x = 0$ and $y = 2$

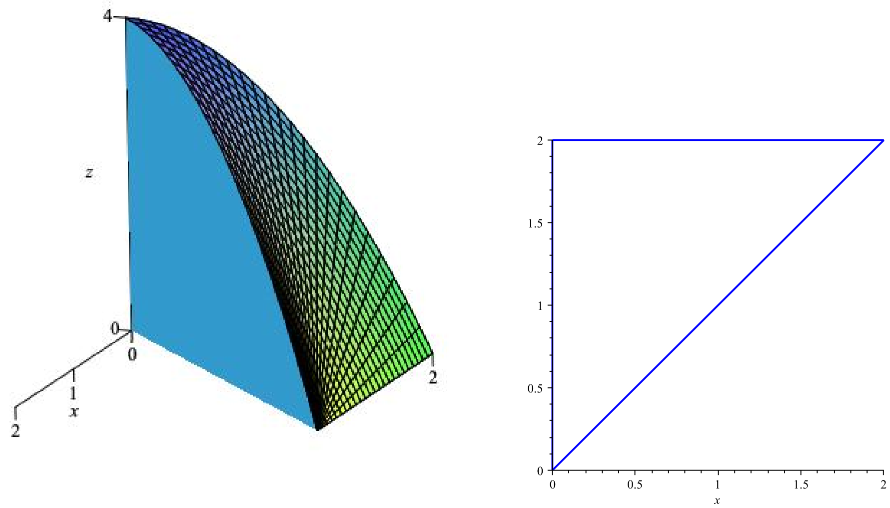


Figure 3: Volume and region of integration

$$V = \int_0^2 \int_x^2 (4 - y^2) dy dx \tag{5}$$

or

$$V = \int_0^2 \int_0^y (4 - y^2) dx dy \tag{6}$$

Volume

$$\begin{aligned} V &= \int_0^2 \int_0^y (4 - y^2) dx dy \\ &= \int_0^2 4x - xy^2 \Big|_0^y dy \\ &= \int_0^2 4y - y^3 \Big|_0^y dy \\ &= 2y^2 - \frac{1}{4}y^4 \Big|_0^2 = 8 - 4 = 4 \end{aligned} \tag{7}$$

Example 3. pg. 987, #24 Find the volume under the plane $z = 4 - x - y$ aon the region bound by $y = x$, $x = 0$ and $y = 2$

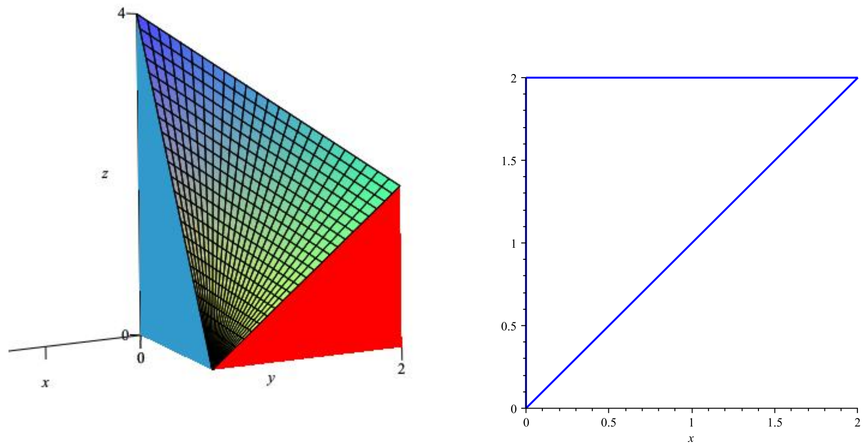


Figure 4: Region of integration

$$V = \int_0^2 \int_x^2 (4 - x - y) dy dx \tag{8}$$

or

$$V = \int_0^2 \int_0^y (4 - x - y) dx dy \tag{9}$$

Example 4. Find the volume under the paraboloid $z = 2 - x^2 - y^2$ and inside the cylinder $x^2 + y^2 = 1$, for $z \geq 0$

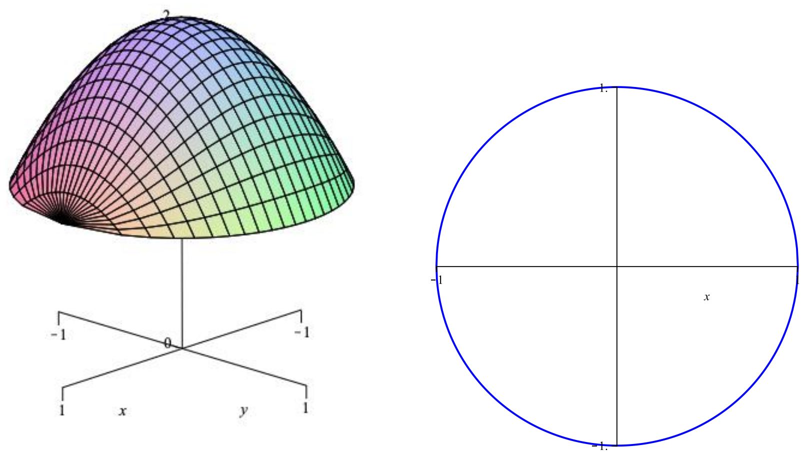


Figure 5: Region of integration

$$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2) dy dx \tag{10}$$

or

$$V = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (2 - x^2 - y^2) dx dy \tag{11}$$

Due to symmetry

$$\begin{aligned} V &= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} (2 - x^2 - y^2) dy dx \\ &= 4 \int_0^1 2y - x^2 y - \frac{2}{3} y^3 \Big|_0^{\sqrt{1-x^2}} dx \\ &= 4 \int_0^1 2\sqrt{1-x^2} - x^2\sqrt{1-x^2} - \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 dx \quad (trig\ sub\ x = \sin\theta) \\ &= \frac{3\pi}{2} \end{aligned} \tag{12}$$

Polar Coordinates

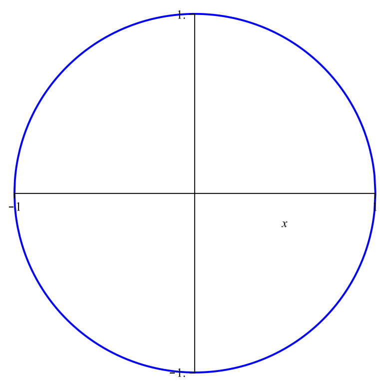
In calculus 2 we introduced polar coordinates where

$$x = r\cos\theta, \quad y = r\sin\theta \tag{13}$$

and

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x} \tag{14}$$

Let's talk about sweeping out the region



We see that

$$r = 0 \rightarrow 1, \quad \theta = 0 \rightarrow 2\pi \tag{15}$$

so what about this integral from the last example

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2 - x^2 - y^2) dydx. \tag{16}$$

How would it change if we used r and θ instead of x and y? Would it becomes easier?

What about the volume under the half sphere $z = \sqrt{1 - x^2 - y^2}$

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}\, dydx. \tag{17}$$