Math 1497 Calc 2

Test # 5 Ratio Test

consider $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges geometric

$r = 1/2$

but what about $\sum_{n=1}^{\infty} \frac{n}{2^n}$

let's consider term way out in series

$n = 100$ $\quad a_{100} = \frac{100}{2^{100}}$ $\quad a_{101} = \frac{101}{2^{101}}$

$= 101$

Note $\quad \frac{a_{101}}{a_{100}} = \frac{\frac{101}{2^{101}}}{\frac{100}{2^{100}}} = \frac{1}{2} \cdot \frac{101}{100}$

$n = 1000$

1001

$$\frac{a_{1001}}{a_{1000}} = \frac{\frac{1001}{2^{1001}}}{\frac{1000}{2^{1000}}} = \frac{1}{2} \frac{1001}{1000}$$

$n \to \infty$ $\quad \frac{a_{n+1}}{a_n} \to \frac{1}{2}$

Ratio Test

if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ (#

if $L < 1$ series conv
$L > 1$ series div
$L = 1$ no conclusion

so $\sum \frac{n}{2^n}$ $\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{1}{2} \frac{n+1}{n}$

$= \frac{1}{2} < 1$

so series conv

Ex2 $\sum_{n=1}^{\infty} \frac{1}{n!}$

DCT $\frac{1}{2} \le \frac{1}{2},$ $\frac{1}{2\cdot 3} \le \frac{1}{2\cdot 2}$ $\frac{1}{2\cdot 3\cdot 4} \le \frac{1}{2\cdot 2\cdot 2}$

$\frac{1}{n!} \le \frac{1}{2^{n-1}}$ can use DCT

$$a_n = \frac{1}{n!} \qquad a_{n+1} = \frac{1}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)\cancel{n!}}}{\frac{1}{\cancel{n!}}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1 \quad \text{series div}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

$$a_n = \frac{(2n)!}{(n!)^2} \qquad a_{n+1} = \frac{(2(n+1))!}{[(n+1)!]^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)!}{(n+1)!^2} \cdot \frac{n!^2}{(2n)!}$$

$$= \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)n!\,(n+1)n!} \cdot \frac{\cancel{n!}}{\cancel{(2n)!}}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$= \lim_{n\to\infty} \frac{2\cancel{(n+1)}(2n+1)}{\cancel{(n+1)}(n+1)} = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4 > 1$$

so by ratio test series converges

Good for powers $2^n, 3^n$ & factorials $n!$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{2^n - 1}{3^n}$

$$\lim_{n\to\infty} \frac{2^{n+1}-1}{3^{n+1}} \Big/ \frac{2^n-1}{3^n} = \lim_{n\to\infty} \frac{2^{n+1}-1}{2^n-1} \cdot \frac{3^n}{3^{n+1}}$$

$$= \frac{1}{3} \lim_{n\to\infty} \frac{2 - \frac{1}{2^n}}{1 - \frac{1}{2^n}} = \frac{2}{3} < 1$$

So conv by ratio test

could LCT w/ $\sum \left(\frac{2}{3}\right)^n$

DCT w/ $\sum \left(\frac{2}{3}\right)^n$ $\qquad \frac{2^n-1}{3^n} < \frac{2^n}{3^n}$ Easiest!

Math 1497 Calc 2

Test #6 Root Test

$$\sum_{n=1}^{\infty} a_n \qquad a_n \geq 0$$

Suppose $\lim_{n\to\infty} \sqrt[n]{a_n} = L$

$L < 1$ series conv

$L > 1$ series div

$L = 1$ no conclusion

Ex: $\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$ ← power

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{2}+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{2}+\frac{1}{n} = \frac{1}{2} < 1$$

So by Root test series conv

Ex2 $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

use L'H

$$\lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e > 1$$

So the series diverge by root test

Ex3 $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$\lim_{n\to\infty} \frac{n^{1/n}}{\ln n}$$

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \infty^0 \quad \text{L'H}$$

$$\lim_{n\to\infty} e^{\ln n^{\frac{1}{n}}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n}$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty} \overset{\text{LH}}{\quad} \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\text{so } \lim_{n\to\infty} n^{\frac{1}{n}} = e^0 = 1$$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{n}}}{\ln n} \to \text{"}\frac{1}{\infty}\text{"}$$

$$= 0 < 1$$

So series conv. by root test.