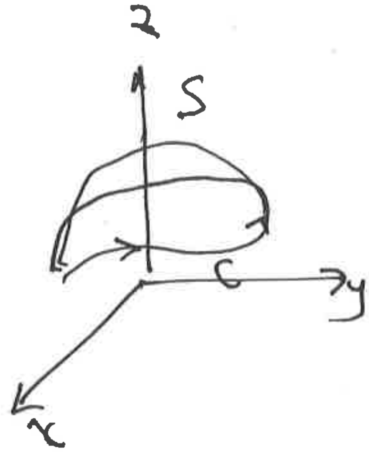
Math 2471 – Calc 3

Stokes' Th^m

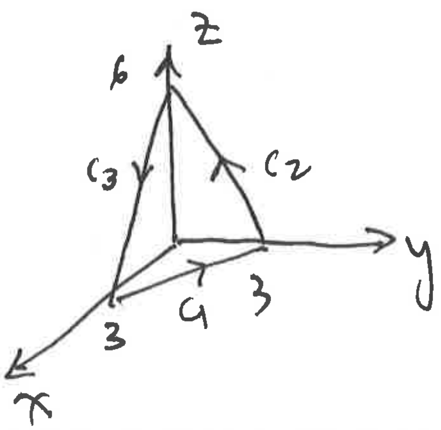
Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise smooth closed boundary C. Suppose $\vec{F}$ is a vector field with cont^s first partial derivatives. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS$$



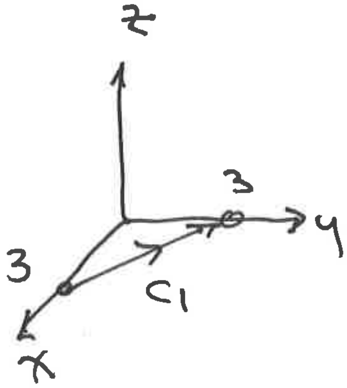
Note: Oriented means walk along C and the surface is on your left

Ex 1 $\vec{F} = \langle -y^2, z, x\rangle$ & S is the plane given by $2x+2y+z=6$



to calculate the line integral we have 3 C's

$\underline{\underline{C_1}}$:



$(3,0,0) \rightarrow (0,3,0)$

so $\bar{r} = \langle -3, 3, 0 \rangle$

$x = 3 - 3t \qquad dx = -3\,dt \qquad 0 \le t \le 1$

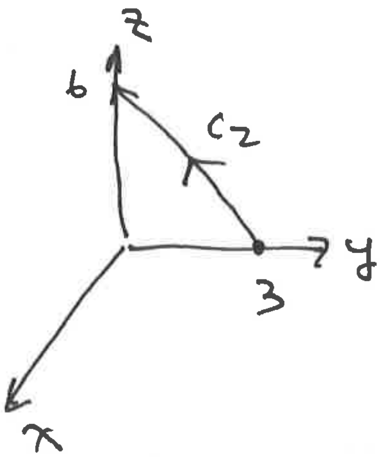
$y = 0 + 3t \qquad dy = 3\,dt$

$z = 0 + 0t \qquad dz = 0$

so $\displaystyle\int_C -y^2\,dx + z\,dy + x\,dz = \int_0^1 -(3t)^2(-3\,dt) + 0 + 0$

$\displaystyle = \int_0^1 27t^2\,dt = 9t^3\Big|_0^1 = 9$

$\underline{\underline{C_2}}$:



$(0,3,0) \rightarrow (0,0,6)$

$\bar{F} = \langle 0, -3, 6 \rangle$

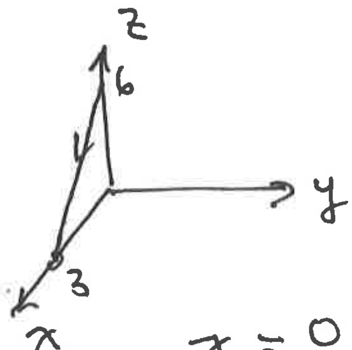
$x = 0 + 0t \qquad 0 \le t \le 1$

$y = 3 - 3t$

$z = 0 + 6t$

$\displaystyle\int_C \underbrace{-y^2\,dx}_{0} + z\,dy + \underbrace{x\,dz}_{0} = \int_0^1 6t(-3\,dt) = -\frac{18t^2}{2}\Big|_0^1 = -9$

C_3:



$(0,0,6) \to (3,0,0)$

$\bar{r} = \langle 3, 0, -6 \rangle$

$x = 0 + 3t$
$y = 0 + 0t$
$z = 6 - 6t$

$0 \le t \le 1$

$$\int_C \underbrace{-y^2 dx}_{0} + \underbrace{z\,dy}_{0} + x\,dz \to \int_0^1 3t(-6\,dt) = -18\int_0^1 t\,dt$$
$$= -9$$

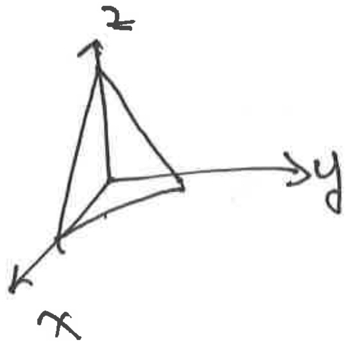
so $\int_C P\,dx + Q\,dy + R\,dz = 9 - 9 - 9 = -9$

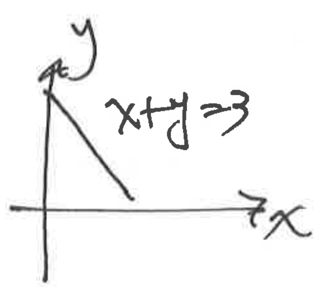
<u>Part 2</u>

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = \langle -1, -1, 2y \rangle$$

$\vec{n} = \langle 2, 2, 1 \rangle \qquad \hat{n} = \dfrac{\langle 2, 2, 1 \rangle}{3}$

if $z = 6 - 2x - 2y \qquad dS = \sqrt{1 + 4 + 4}\, dA = 3\,dA$



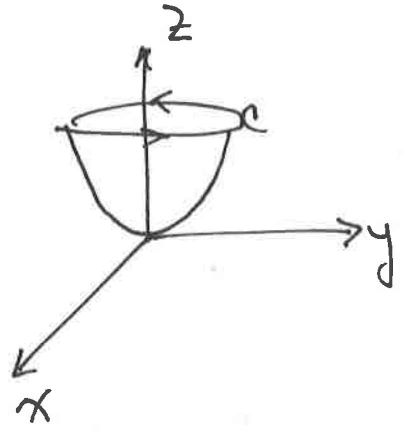


$$\int_0^3 \int_0^{3-x} \langle -1, -1, 2y \rangle \cdot \frac{\langle 2, 2, 1 \rangle}{\cancel{3}} \cancel{3}\, dA$$

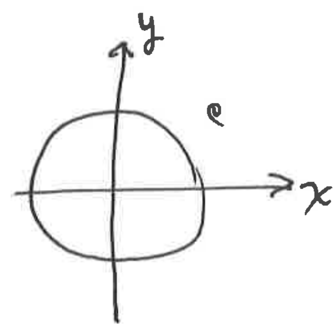
$$\int_0^3 \int_0^{3-x} (-2 - 2 + 2y)\,dy\,dx = -9$$

Ex2 S: top of the paraboloid given by

$z = x^2+y^2$, $z=1$, $\vec{F} = \langle z, x, y+2z \rangle$



top view



So we want to evaluate

$$\int z\,dx + x\,dy + (y+2z)\,dz$$

parametrically, the curve is given by

$x = \cos t$, $y = \sin t$, $z = 1$ $\quad 0 \le t \le 2\pi$

$dx = -\sin t\,dt \quad dy = \cos t\,dt \quad dz = 0$

So line $\int$ becomes

$$\int_0^{2\pi} 1\cdot(-\sin t)\,dt + \cos^2 t\,dt + 0$$

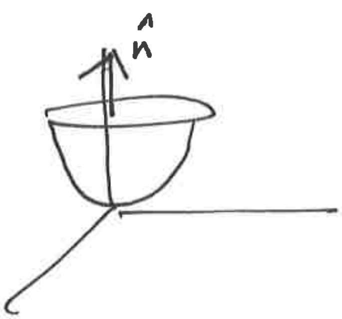
$= \pi$

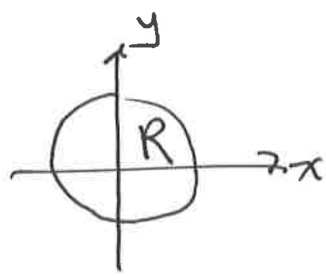
For part 2 of the Th^{m}

$$\overline{\nabla}\times\overline{F} = \begin{vmatrix} \vec{i} & \vec{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y+2z \end{vmatrix}$$

$$= \langle 1, 1, 1 \rangle$$

Next, $\hat{n} = \langle 0, 0, 1 \rangle$





so $$\iint_S \overline{\nabla}\times\overline{F}\cdot\hat{n}\, ds = \iint_R \langle 1,1,1\rangle \cdot \langle 0,0,1\rangle\, dA$$

$$= \iint_R dA = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi$$ same//