Math 6345 - Adv. ODE's

Consider
$$\dot{x} = y + kx(x^2+y^2)$$
$$\dot{y} = -x + ky(x^2+y^2)$$

For the critical pts set $\dot{x}=0$, $\dot{y}=0$

so
$$y + kx(x^2+y^2) = 0$$
$$-x + ky(x^2+y^2) = 0$$

$$\Rightarrow \left.\begin{array}{l} y^2 + kxy(x^2+y^2) = 0 \\ -x^2 + kxy(x^2+y^2) = 0 \end{array}\right\} \quad x^2+y^2 = 0 \Rightarrow x = y = 0$$

So $(0,0)$ is the critical pt.

<u>Linearized System</u>

$$D_x f = \begin{pmatrix} k(3x^2+y^2) & 1+2kxy \\ -1+2kxy & k(x^2+3y^2) \end{pmatrix}$$

& $$D_x f\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so the linearized system is

$$\dot{\bar{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{x}$$

Eigenvalues are: $\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = 0 \quad \lambda^2 + 1 = 0$

$$\lambda = \pm i$$

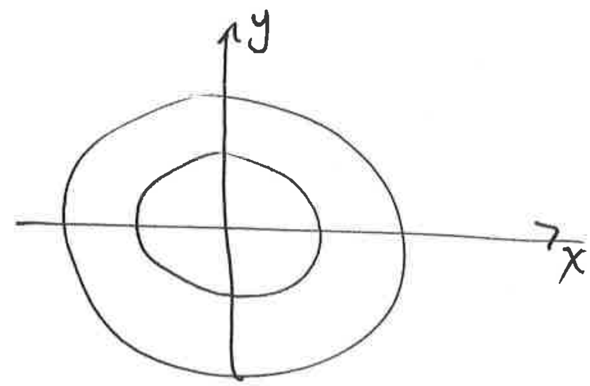
so the linear system predicts a center

In fact the phase plane is

$$\dot{x} = y, \quad \dot{y} = -x$$

$$\Rightarrow \quad x\dot{x} + y\dot{y} = xy - xy = 0$$

$$\Rightarrow \quad x^2 + y^2 = c$$



However, if we switch to polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{or} \quad r^2 = x^2 + y^2, \quad \theta = \tan^{-1} y/x$$

so $\quad 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$

$$= 2x\left(y + kx(x^2+y^2)\right) + 2y\left(-x + ky(x^2+y^2)\right)$$

$$= 2\cancel{x}y + 2kx^2(x^2+y^2) - 2\cancel{x}y + 2ky^2(x^2+y^2)$$

$$= 2k(x^2+y^2)^2$$

$$\Rightarrow \quad r\dot{r} = kr^4$$

$$\Rightarrow \quad \dot{r} = kr^3$$

Further, $\dot{\theta} = \dfrac{x\dot{y} - y\dot{x}}{x^2+y^2}$

$$= \frac{x(-x + ky(x^2+y^2)) - y(y + kx(x^2+y^2))}{x^2+y^2}$$

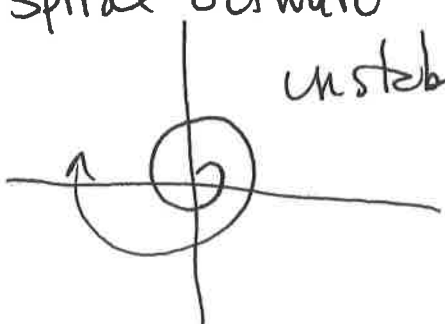
$$= \frac{-x^2 + \cancel{kxy(x^2+y^2)} - y^2 - \cancel{kxy(x^2+y^2)}}{x^2+y^2}$$

$$= -1$$

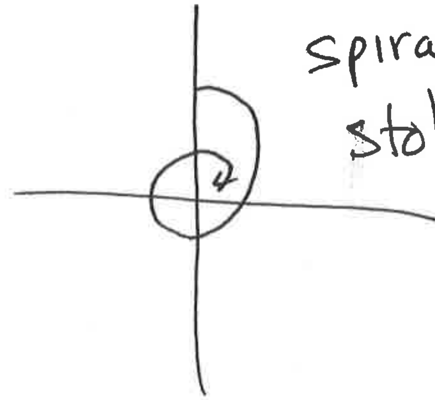
so the actual system reduces (simplifies) to

$$\dot{r} = kr^3, \quad \dot{\theta} = -1$$

if $k > 0$ $\dot{r} > 0$ spiral outward unstable



$k < 0$ $\dot{r} < 0$ spiral inward stable



Ex 2 $\quad \dot{x} = -x - 2y^2 \qquad$ CP $\quad -x-2y^2=0$

$\qquad\quad \dot{y} = xy - y^3 \qquad\qquad\quad y(x-y^2)=0$

From the second $y=0 \Rightarrow x=0$ or $\left.\begin{array}{l} x-y^2=0 \\ x+2y^2=0 \end{array}\right\}$ $(0,0)$

Linearized System

$$D_x f = \begin{pmatrix} -1 & -4y \\ y & x-3y^2 \end{pmatrix}$$

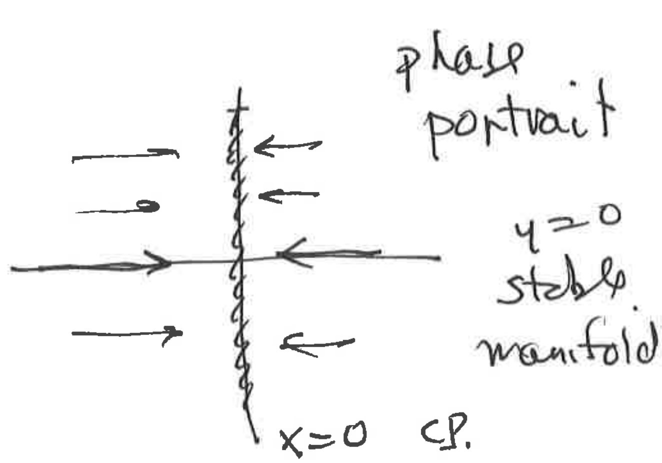
$$D_x f\big|_{CP} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

so Linear Sys. $\quad \dot{\bar{x}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \bar{x}$

Eigenvalues $\quad \begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda \end{vmatrix} = 0 \quad \lambda(\lambda+1)=0 \quad \lambda = 0, -1$

$\lambda = 0 \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bar{u} = \bar{0} \Rightarrow \bar{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{0}$

$\lambda = -1 \quad \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bar{u} = \bar{0} \Rightarrow \bar{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$



Sol^n $\quad \bar{x} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$

So in these examples, the linear system <u>does not</u> predict the actual stability of the non linear system.

<u>Def</u>

<u>Def^n</u> Hyperbolic critical pt

A critical pt is said to be hyperbolic if the eigenvalue of the Jacobian matrix $D_x f$ has nonzero real part.

<u>Hartman - Grobman - Th^m</u>

If a nonlinear system has a ~~no~~ hyperbolic critical pt $x = x^*$ then the stability of this critical pt is determined by the linear system at the critical pt.

So we need another way to predict stability when the critical pt is nonhyperbolic

Def^{n} Suppose that $V(x,y)$ is cont^{s} in a neighborhood D of $(0,0)$ and that $V(0,0)=0$

If $V(x,y) \geq 0$ for $(x,y) \in D$ then V is positive semi definite

If $V(x,y) > 0$ for $(x,y) \in D \setminus (0,0)$ then V is positive definite

If $V(x,y) \leq 0$ for $(x,y) \in D$, V negative semi definite

If $V(x,y) < 0$ for $(x,y) \in D \setminus (0,0)$ V neg. definite

Lyapunov Stability

Let $V(x,y)$ be cont^{s} differentiable function (pos. def.) on a domain D that contains the origin.

The derivative of V along the trajectory of

$$\dot{x} = f(x,y) \quad \dot{y} = g(x,y)$$

is $\dot{V} = V_x \dot{x} + V_y \dot{y} = V_x f + V_y g$

If $\dot{V} \leq 0$ in D $(0,0)$ is stable

$\dot{V} < 0$ in $D \setminus (0,0)$ then $(0,0)$ is Asy. Stable

So the trick is coming up with these Lyopunov functions

Ex 1 $\quad \dot{x} = y + kx(x^2+y^2)$

$\dot{y} = -x + ky(x^2+y^2)$

Consider $\quad V = x^2+y^2 \quad$ (this was r^2)

Now $\quad V(0,0) = 0$ & $V(x,y) > 0$ if $(x,y) \neq (0,0)$

so $\quad \dot{V} = 2x\dot{x} + 2y\dot{y}$

$= 2x(y+kx(x^2+y^2)) + 2y(-x+ky(x^2+y^2))$

$= 2k(x^2+y^2)^2$

So $\dot{V} > 0$ if $k > 0$ so Asy. Stable

$\dot{V} < 0$ if $k < 0$ so unstable

We saw this already

Ex 2 $\quad \dot{x} = -x - 2y^2$

$\dot{y} = xy - y^3$

Try $V = x^2 + y^2$ (already shown to be Lypunov)

$\dot{V} = 2x\dot{x} + 2y\dot{y}$

$= 2x(-x - 2y^2) + 2y(xy - y^3)$

$= -2x^2 - 4xy^2 + 2xy^2 - 2y^4$

$= -2x^2 - 2xy^2 - 2y^4$

↑ close - would like this to be gone

Next, try $V = x^2 + ay^2$ let's find a

if $a > 0$ then V is Lyp

$\dot{V} = 2x\dot{x} + 2ay\dot{y}$

$= 2x(-x - 2y^2) + 2ay(xy - y^3)$

$= -2x^2 - 4xy^2 + 2axy^2 - 2ay^4$ pick $a = 2$

$\dot{V} = -2x^2 - 4y^4 < 0$ so $(0,0)$ is Asy. stable