Toronto Math Circles: Full Solutions Mathematics Competition

Junior

Saturday December 5, 2015

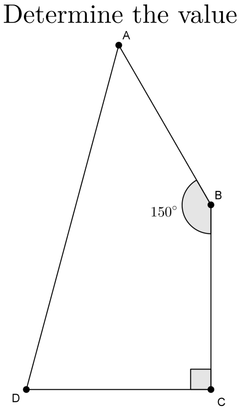
1:00 pm - 3:00 pm

Each problem is graded on a basis of 0 to 10 points. All the necessary work to justify an answer must be shown clearly to obtain full credit. Some partial credit may be given, but only when a contestant has shown significant and substantial progress toward a solution.

1. A die is said to be regular if the number on its faces are $1, 2, 3, 4, 5, 6$. A die is said to be fair if the probability of landing on any face is the same. Two regular fair dice are rolled. What is the most likely greatest common divisor of the two numbers? What is the probability of this occurring?
   Note: Greatest common divisor is the same thing as greatest common factor.

2. Determine the maximum number of dots that can be placed in a 5×6 grid such that no two dots are in the same row, column or diagonal.
   Note: 5×6 means 5 rows and 6 columns.

3. Let $ABCD$ be a quadrilateral such that $AB = BC = CD$ and $\angle ABC = 150°$ and $\angle BCD = 90°$. Determine the value of $\angle CDA$.



4. Consider the list of numbers

$$1, 12, 123, 1234, \ldots, 123456789, 1234567890, 12345678901, 123456789012, \ldots$$

   Find the position of all number in this list that are divisible by 90.
   Note: 1 is at position 1 and 12 is at position 2 and etc.

5. Let x and y be two real numbers that satisfy the equation

$$x^2 + y^2 + 12xy = 2016$$

   Determine the maximum possible value of xy.

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1. Consider the list of numbers

$$1, 12, 123, 1234, \ldots, 123456789, 1234567890, 12345678901, 123456789012, \ldots$$

   Find the position of all number in this list that are divisible by 90.
   Note: 1 is at position 1 and 12 is at position 2 and etc.

2. Compute the area of the region

$$\{(x,y) \in \mathbb{R}^2 | (x^2+y^2)^2 \leq 4x^2, |x|+|y| \geq 1\}$$

3. Let x_n be a sequence of real numbers such that $x_1 = 1$ and $0 \leq x_k \leq 2x_{k-1}$ for $k = 2, 3, \ldots$. Let m be a non-negative integer, determine the maximum value of

$$x_1 - x_2 + x_3 - x_4 + \cdots + x_{2m+1} - x_{2m+2}$$

   Express your answer in simplest form.

4. A die is said to be fair if the probability of landing on any face is the same. Let A and B be two 6 sided fair dice with a positive integer on each face. Suppose that the sum of their rolls gives the following probability table

Sum: ($n =$)	2	3	4	5	6	7	8	9	10	11	12
Probability of n	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

   Find all possible combinations of dice, A and B, such that the set of numbers on die A is different from the set of numbers on die B.

5. Let x_n be a sequence of positive real numbers such that

$$x_n = \sqrt{\frac{6x_{n-1}^3 x_{n-3} - 8x_{n-2}^3 x_{n-1}}{x_{n-2}x_{n-3}}}$$

   and $x_1^2 = 1$, $x_2^2 = 2$ and $x_3^2 = 12\sqrt{2}$. Prove that for any positive integer n,

$$x_n \prod_{k=1}^{n} x_k$$

   is an integer and divisible by n.