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UNSTEADY UNIDIRECTIONAL FLOWS OF SECOND GRADE FLUIDS IN DOMAINS WITH HEATED BOUNDARIES

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Abstract—The thermal convection of a second grade fluid is studied within the context of unidirectional flows. Stokes' first problem in finite and infinite domains as well as pulsating Poiseuille flow in a pipe are analyzed. Due to the special nature of the flow, both the balance of linear momentum and the energy equation can be solved exactly.

1. INTRODUCTION

The determination of the temperature distribution within a layer of liquid when the internal friction is not negligible is of utmost importance in lubrication problems and has relevance to applications that involve periodic motions of the boundaries, fluctuations of the pressure gradient or both (e.g., the motion of a piston inside the cylinder of an engine, the flow of a fluid in a pipeline operated by a volumetric pump, etc.).

While there have been numerous studies of thermal convection in non-Newtonian fluids, they have primarily considered steady state problems, particularly of power-law fluids. Here, our interest lies in studying unsteady flows of non-Newtonian fluids in domains wherein the boundary is heated. Such problems have been addressed in the case of the classical linearly viscous fluid; Faggiani *et al.* [1] considered periodic Couette flow and Uchida [2] periodic Poiseuille flow in a pipe. Thermal problems for unsteady motions of viscoelastic fluids have not received much attention. Rajagopal and Na [3] and Szeri and Rajagopal [4] looked at the flow of a non-Newtonian fluid between parallel plates. Rajagopal [5] found exact solutions for a number of unidirectional flows involving a second grade fluid under isothermal conditions. Here we extend that analysis by allowing the boundary of the flow domain to be heated: we derive the energy equation for second grade fluids valid for some unidirectional motions and obtain exact solutions for some flows and geometric configurations of practical interest.

2. GOVERNING EQUATIONS

The incompressible fluid of second grade is characterized by the following constitutive equation [5]:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{2.1}$$

where μ is the coefficient of viscosity, α_1 and α_2 are normal stress moduli, $-pI$ denotes the indeterminate spherical stress and A_1 and A_2 are the kinematic tensors defined through

$$A_1 = (\operatorname{grad} v) + (\operatorname{grad} v)^T, \tag{2.2a}$$

and

$$A_2 = \frac{\mathrm{d}}{\mathrm{d}t} A_1 + A_1(\operatorname{grad} v) + (\operatorname{grad} v)^T A_1. \tag{2.2b}$$

Here v is the velocity, grad the gradient operator and (d/dt) the material time derivative.

Contributed by K. R. Rajagopal.

If we substitute the stress T in the balance of linear momentum

$$\operatorname{div} T + \rho b = \rho \frac{\mathrm{d}v}{\mathrm{d}t}, \tag{2.3}$$

we obtain, in the case of a conservative body force field $b = -\operatorname{grad}\phi$

$$\mu\Delta v + \alpha_1 \Delta v_t + \alpha_1(\Delta w \times v) + (\alpha_1 + \alpha_2)\{A_1 \Delta v + 2\operatorname{div}[(\operatorname{grad} v)(\operatorname{grad} v)^T]\} - \rho(w \times v) - \rho v_t = \operatorname{grad} P, \tag{2.4}$$

where

$$P = p - \alpha_1(v \cdot \Delta v) - \frac{(2\alpha_1 + \alpha_2)}{4}|A_1|^2 + \frac{1}{2}\rho|v|^2 + \rho\phi, \tag{2.5}$$

and Δ is the Laplacian, the subscript t indicates partial differentiation with respect to time, $|A_1|$ the trace norm of A_1 and

$$w = \operatorname{curl} v. \tag{2.6}$$

We have assumed that the fluid is incompressible which implies that it can undergo only isochoric motions, therefore

$$\operatorname{div} v = 0. \tag{2.7}$$

The energy equation for forced convection is

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = T \cdot L - \operatorname{div} q + \rho r, \tag{2.8}$$

where ε is the specific internal energy, L the gradient of velocity, q the heat flux vector and r the radiant heating. It can be shown (cf. [3]) that, for a second grade fluid,

$$T \cdot L = \frac{\mu}{2}|A_1|^2 + \frac{\alpha_1}{4}\frac{\mathrm{d}}{\mathrm{d}t}|A_1|^2 + \frac{\alpha_1 + \alpha_2}{4} tr A_1^3. \tag{2.9}$$

Also, for the model (2.1) to be compatible with thermodynamics, the specific Helmholtz free energy which characterizes the fluid has to take the form

$$\psi = \psi(\theta, A_1, A_2, A_3) = \bar{\psi}(\theta, 0) + \frac{\alpha_1}{4\rho}|A_1|^2, \tag{2.10}$$

and the specific entropy must be defined through

$$\eta = -\bar{\psi}_\theta, \tag{2.11}$$

where the subscript denotes partial differentiation with respect to that variable. Since the specific internal energy is related to the Helmholtz free energy through

$$\varepsilon = \psi + \theta\eta, \tag{2.12}$$

it follows from (2.10)–(2.12) that

$$\varepsilon = \hat{\phi}(\theta) + \frac{\alpha_1}{4\rho}|A_1|^2 - \theta\hat{\phi}_\theta, \tag{2.13}$$

where

$$\hat{\phi}(\theta) = \bar{\psi}(\theta, 0). \tag{2.14}$$

Thus,

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = \rho\left\{\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\phi}(\theta) - \theta\hat{\phi}_\theta) + \frac{\alpha_1}{4\rho}\frac{\mathrm{d}}{\mathrm{d}t}|A_1|^2\right\}. \tag{2.15}$$

Next, note that (2.11)–(2.12) imply that

$$\varepsilon_\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}(\hat{\phi} - \theta\hat{\phi}_\theta) = -\theta\hat{\phi}_{\theta\theta} \equiv c, \tag{2.16}$$

where c is called the specific heat. Thus

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = c\frac{\mathrm{d}\theta}{\mathrm{d}t}. \tag{2.17}$$

By virtue of Fourier's law of heat conduction

$$q = -k \operatorname{grad} T,$$

where k is the conductivity, which is assumed to be constant, and T the temperature. The energy equation becomes

$$\rho c\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mu}{2}|A_1|^2 + k\Delta T + \rho r, \tag{2.18}$$

which is formally identical to the energy equation for a Navier–Stokes fluid. Of course the solution of (2.18) yields a temperature field that is different from the Newtonian case because the velocity distribution resulting from the balance of linear momentum is different.

We will consider unidirectional flows of the form

$$v = v(y, t)i,$$

(i denotes a unit vector along the x-coordinate direction and y the coordinate along a direction perpendicular to i) and a temperature field of the form

$$T = T(y, t).$$

3. FLOW DUE TO A HEATED RIGID PLATE OSCILLATING IN ITS OWN PLANE

Suppose that a second grade fluid occupies the space above a plate oscillating with speed $U\cos\omega t$. Let T_0 and T_∞ denote respectively the temperatures of the plate and of the fluid at infinity. Rajagopal [5] showed that the velocity is

$$v(y, t) = U\mathrm{e}^{-my}\cos(\omega t - ny), \tag{3.1}$$

where

$$m^2 = \frac{1}{2}\frac{\rho\omega}{[\mu^2 + (\alpha_1\omega)^2]}\{[\mu^2 + (\alpha_1\omega)^2]^{1/2} + \alpha_1\omega\}, \tag{3.2}$$

and

$$n^2 = \frac{1}{2}\frac{\rho\omega}{[\mu^2 + (\alpha_1\omega)^2]}\{[\mu^2 + (\alpha_1\omega)^2]^{1/2} - \alpha_1\omega\}. \tag{3.3}$$

The energy equation as resulting from (2.18) is, in dimensionless form,

$$\rho c\frac{\partial\theta}{\partial t} = \frac{\mu}{T_0 - T_\infty}\left(\frac{\partial v}{\partial y}\right)^2 + k\frac{\partial^2\theta}{\partial y^2}, \tag{3.4}$$

where

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}. \tag{3.5}$$

Now relationship (3.1) can be rewritten as

$$v(y, t) = \mathrm{Re}\{U\mathrm{e}^{i\omega t}\exp[-y(m + in)]\}, \tag{3.6}$$

where Re denotes the real part of the expression in parentheses. Equation (3.4) will be solved following the complex temperature method (cf. [6]): assuming a temperature field of the form

$$\theta(y, t) = \theta(y)\mathrm{e}^{2i\omega t}, \tag{3.7}$$

and substituting it into the energy equation, we find that $\theta(y)$ needs to satisfy the linear ordinary differential equation

$$2i\omega\rho c\theta - k\theta'' = Af(y), \tag{3.8}$$

subject to the boundary conditions

$$\theta(0) = 1, \qquad \theta(\infty) = 0, \tag{3.9}$$

where

$$A = \frac{\mu[U(m + in)]^2}{T_0 - T_\infty}, \tag{3.10}$$

and

$$f(y) = \exp[-2y(m + in)]. \tag{3.11}$$

Prime denotes derivative with respect to y. It is easy to show that

$$\theta(y, t) = \exp\left(-\frac{qy}{\sqrt{2}}\right)\left[(1 - D)\cos\left(2\omega t - \frac{qy}{\sqrt{2}}\right) + E\sin\left(2\omega t - \frac{qy}{\sqrt{2}}\right)\right]$$
$$+ \exp(-2my)[D\cos 2(\omega - n)t - E\sin 2(\omega - n)t] \tag{3.12}$$

where

$$q = \sqrt{\frac{2\rho c\omega}{k}}, \tag{3.13}$$

$$D = \frac{\mu U^2}{T_0 - T_\infty}\frac{4(m^2 - n^2)^2 - 2mn(q^2 - 8mn)}{(q^2 - 8mn)^2 + 16(m^2 - n^2)^2}, \tag{3.14}$$

$$E = \frac{\mu U^2}{T_0 - T_\infty}\frac{q^2(m^2 - n^2)}{(q^2 - 8mn)^2 + 16(m^2 - n^2)^2}. \tag{3.15}$$

4. FLOW BETWEEN TWO INFINITE PARALLEL PLATES ONE OF WHICH IS OSCILLATING

Suppose that a second grade fluid occupies the slot between two parallel plates at temperature T_1 and T_2. If d denotes the distance between the plates and the plate at $y = 0$ oscillates with velocity $U\cos\omega t$, Rajagopal [5] showed that the following solution exists:

$$v(y, t) = \mathrm{Re}\left\{U\frac{\sinh\gamma(d - y)}{\sinh\gamma d}e^{i\omega t}\right\}, \tag{4.1}$$

where

$$\gamma^2 = \frac{\rho\omega(\alpha_1\omega + i\mu)}{[\mu^2 + (\alpha_1\omega)^2]}.$$

Upon substitution of this expression into the energy equation and assuming the same temperature field as in (3.7), this is determined as

$$\theta(y, t) = \mathrm{Re}\left\{\left(D\cosh py + E\sinh py - \frac{A}{2p^2}\left[1 + \frac{p^2}{p^2 - 4\gamma^2}\cosh 2\gamma(d - y)\right] + \frac{y}{d}\right)e^{2i\omega t}\right\}, \tag{4.2}$$

where

$$p = \sqrt{\frac{2i\rho c\omega}{k}}, \quad A = \frac{1}{T_1 - T_2}\left(\frac{\gamma U}{\sinh\gamma d}\right)^2, \quad D = \frac{A}{2p^2}\left(1 + \frac{p^2\cosh 2\gamma d}{p^2 - 4\gamma^2}\right),$$

$$E = \frac{A}{2p^2\sinh pd}\left[1 + \frac{p^2}{p^2 - 4\gamma^2} - \left(1 + \frac{p^2\cosh 2\gamma d}{p^2 - 4\gamma^2}\right)\cosh pd\right].$$

5. TIME PERIODIC POISEUILLE FLOW IN AN ANNULUS AND A PIPE

Suppose that a second grade fluid fills an annulus of internal radius r_0 and external radius r_1 and that the pressure gradient along the x-direction is given by

$$\frac{\partial p}{\partial x} = -\rho Q_0 \cos \omega t. \tag{5.1}$$

It can be shown (cf. [7]) that the momentum equation is

$$\mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) - \rho\frac{\partial w}{\partial t} = \frac{\partial p}{\partial x}, \tag{5.2}$$

with the boundary conditions

$$w(r_0, t) = w(r_1, t) = 0.$$

In complex form expression (5.1) can be written as

$$\frac{\partial p}{\partial x} = -\rho Q_0 \operatorname{Re}\{e^{i\omega}\}. \tag{5.3}$$

If we seek a solution of the form

$$w = f(r)\, e^{i\omega t}, \tag{5.4}$$

the function $f(r)$ has to satisfy the ordinary differential equation

$$f'' + \frac{f'}{r} - \frac{i\rho\alpha_1\omega}{\mu + i\alpha_1\omega} f + \frac{\rho Q_0}{\mu + i\alpha_1\omega} = 0, \tag{5.5}$$

subject to boundary conditions

$$f(r_0) = f(r_1) = 0. \tag{5.6}$$

We find that

$$f(r) = \frac{Q_0}{i\alpha_1\omega}\left[1 - \frac{(I_0(\gamma r_0) - I_0(\gamma r_1))\, K_0(\gamma r) - (K_0(\gamma r_0) - (K_0(\gamma r_1))\, I_0(\gamma r)}{K_0(\gamma r_1)\, I_0(\gamma r_0) - K_0(\gamma r_0)\, I_0(\gamma r_1)}\right], \tag{5.7}$$

where

$$\gamma = \sqrt{\frac{i\rho\alpha_1\omega}{\mu + i\alpha_1\omega}}.$$

If $r_0 = 0$, then

$$f(r) = \frac{Q_0}{i\alpha_1\omega}\left[1 - \frac{I_0(\gamma r)}{I_0(\gamma r_1)}\right]. \tag{5.8}$$

The energy equation leads to

$$\rho c\frac{\partial T}{\partial t} = \mu\left(\frac{\partial w}{\partial r}\right)^2 + k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \tag{5.9}$$

Under the boundary conditions

$$T(r_1) = T_1, \qquad T(0) = \text{finite}, \tag{5.10}$$

equation (5.9) yields

$$T = \operatorname{Re}\{C_1 I_0(qr) e^{2i\omega t}\}, \tag{5.11}$$

where

$$C_1 = \mu q\left(\frac{Q_0}{\alpha_1\omega}\right)^2 \int\left(1 + \frac{I_0^2(\gamma r)}{I_0^2(\gamma r_1)} - 2\frac{I_0(\gamma r)}{I_0(\gamma r_1)}\right) K_0(qr)\,dr + c_1, \tag{5.12}$$

c_1 is a constant to be determined from the boundary conditions and q has been defined in the previous section.

6. TIME PERIODIC POISEUILLE FLOW IN A SLOT

The product of three modified Bessel functions makes the analytical computation of the temperature field through (5.10)–(5.12) a daunting task. When the gap between the two cylinders is small in comparison to the average radius the problem may be regarded as plane and the analysis considerably simplified.

If the pressure gradient along the x-direction $\partial p/\partial x$ is given by

$$\frac{\partial p}{\partial x} = -\rho(P_0 + Q_0 \cos \omega t), \tag{6.1}$$

the velocity field is (cf. [5])

$$v(y,t) = \frac{\rho h^2 P_0}{2\mu}[1-(y/h)^2] + \frac{2Q_0}{\omega}\mathrm{Re}\left\{e^{i\omega t}\left[\frac{\cosh \beta(1+i) - \cosh \beta(1+i)y/h}{2i\cosh \beta(1+i)}\right]\right\}, \tag{6.2}$$

where

$$\beta = \frac{1}{\sqrt{2}}\frac{[\rho h^2 \omega/2\mu]^{1/2}}{\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}}\left\{\left\{\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}+1\right\}^{1/2} - \left\{i\left[1+\left(\frac{\alpha_1\omega}{\mu}\right)^2\right]^{1/2}-1\right\}^{1/2}\right\}. \tag{6.3}$$

By taking the derivative of the complex form of (6.2) and substituting it into the energy equation (2.18), we have

$$\rho c\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial y^2} + Ay^2 + B\sinh^2 by \exp(2i\omega t) + Cy\sinh by \exp(i\omega t), \tag{6.4}$$

where

$$A = \frac{\mu}{T_1 - T_2}\left(\frac{\rho h^2 P_0}{2\mu}\right)^2, \tag{6.5}$$

$$B = \frac{\mu}{T_1 - T_2}\left(\frac{Q_0 b}{i\omega \cosh bh}\right)^2, \tag{6.6}$$

$$C = 2\sqrt{AB}, \tag{6.7}$$

and

$$b = \frac{\beta(1+i)}{h}, \tag{6.8}$$

$$T(0) = T_1, \qquad T(h) = T_2.$$

We will invoke the boundary conditions

$$\theta(0,t) = 0, \qquad \theta(h,t) = 1, \tag{6.9}$$

and, by use of the complex temperature method, seek a solution of the form

$$\theta = \theta_1(y) + \theta_2(y)\exp(2i\omega t) + \theta_3(y)\exp(i\omega t). \tag{6.10}$$

Relationships (6.9) and (6.10) imply that

$$\theta_1(0) = \theta_2(0) = \theta_3(0) = \theta_2(h) = \theta_3(h) = 0, \qquad \theta_1(h) = 1. \tag{6.11}$$

Straightforward calculation yields

$$\theta_1(y) = y\left(\frac{1}{h} + \frac{A}{12k}(y^3 - h^3)\right), \tag{6.12}$$

and

$$\theta_2(y) = \frac{B}{2p^2}\left(1 - \frac{p^2\cos 2by}{p^2 - 4b^2}\right) + k_1\cosh py + k_2 \sinh py, \tag{6.13}$$

where

$$k_1 = -\frac{B}{2p^2}\left(1 - \frac{p^2}{p^2 - 4b^2}\right), \tag{6.14}$$

$$k_2 = \frac{B}{2p^2 \sinh ph}\left[\left(1 - \frac{p^2}{p^2 - 4b^2}\right)\cosh ph - \left(1 - \frac{p^2 \cos 2b}{p^2 - 4b^2}\right)\right]. \tag{6.15}$$

Also

$$\theta_3(y) = -\frac{C}{p^2 - b^2}\left(y \sinh by + \frac{2b}{p^2 - b^2}\cosh by\right) + C_1 \sinh py + C_2 \cosh py, \tag{6.16}$$

$$C_2 = \frac{2Cb}{(p^2 - b^2)^2}, \tag{6.17}$$

$$C_1 = \frac{C}{(p^2 - b^2)\sinh ph}\left[\frac{2b}{p^2 - b^2}(\cosh bh - \cosh ph) + h \sinh bh\right]. \tag{6.18}$$

REFERENCES

1. S. Faggiani, F. Nurzia and A. Satta, Velocity and temperature distribution in periodic Couette flow. *Communications of the Third World Congress for the Theory of Machines and Mechanisms*, Kupari, Yugoslavia, Sept. 13–20, 1971.
2. S. Uchida, The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe. *ZAMP* 7, 403–422 (1956).
3. A. Z. Szeri and K. R. Rajagopal, Flow of a non-Newtonian fluid between parallel plates. *Int. J. Non-Linear Mechanics* **20**, 91–101 (1985).
4. K. R. Rajagopal and T. Y. Na, Natural convection flow of a non-Newtonian fluid between two vertical flat plates. *Acta Mech.* **54**, 239–246 (1985).
5. K. R. Rajagopal, A note on unsteady unidirectional flows of a non-Newtonian fluid. *Int. J. Non-Linear Mechanics* **17**, 369–373 (1982).
6. V. S. Arpaci, *Convection Heat Transfer*. Prentice-Hall, Englewood Cliffs, NJ (1984).
7. K. R. Rajagopal, Longitudinal and torsional oscillations of a rod in a second grade fluid. *Acta Mech.* **49**, 281–285 (1983).