Solutionbank C1
Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions
Exercise A, Question 1

Question:

The line L has equation $y = 5 - 2x$.

(a) Show that the point $P(3, -1)$ lies on L.

(b) Find an equation of the line, perpendicular to L, which passes through P. Give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

(a)

For $x = 3$,

$y = 5 - (2 \times 3) = 5 - 6 = -1$

Substitute $x = 3$ into the equation of L .

So $(3 , -1)$ lies on L .

Give a conclusion.

(b)

$y = -2x + 5$

Gradient of L is -2 .

Compare with $y = mx + c$ to find the gradient m

Perpendicular to L , gradient is $\frac{1}{2}$ ($\frac{1}{2} \times -2 = -1$)

For a perpendicular line, the gradient is $-\frac{1}{m}$

$y - (-1) = \frac{1}{2} (x - 3)$

Use $y - y_1 = m (x - x_1)$

$y + 1 = \frac{1}{2}x - \frac{3}{2}$

Multiply by 2

$2y + 2 = x - 3$

$0 = x - 2y - 5$

$x - 2y - 5 = 0$

($a = 1$, $b = -2$, $c = -5$)

This is the required form $ax + by + c = 0$, where a , b and c are integers.

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Algebraic fractions
Exercise A, Question 2

Question:

The points A and B have coordinates $(-2, 1)$ and $(5, 2)$ respectively.

(a) Find, in its simplest surd form, the length AB.

(b) Find an equation of the line through A and B, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

The line through A and B meets the y-axis at the point C.

(c) Find the coordinates of C.

Solution:

(a)

A : $(-2, 1)$, B $(5, 2)$

AB $= \sqrt{(5-(-2))^2 + (2-1)^2}$

$= \sqrt{(7^2 + 1^2)} = \sqrt{50}$

The distance between two points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (Pythagoras's Theorem)

$\sqrt{50} = \sqrt{(25 \times 2)} = 5\sqrt{2}$

Use $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$

AB $= 5\sqrt{2}$

(b)

$m = \frac{2-1}{5-(-2)} = \frac{1}{7}$

Find the gradient of the line, using $m = \frac{y_2 - y_1}{x_2 - x_1}$

$y - 1 = \frac{1}{7}(x - (-2))$

Use $y - y_1 = m(x - x_1)$

$y - 1 = \frac{1}{7}x + \frac{2}{7}$

Multiply by 7

$7y - 7 = x + 2$

$0 = x - 7y + 9$

$x - 7y + 9 = 0$

$(a = 1, b = -7, c = 9)$

This is the required form $ax + by + c = 0$, where a, b and c are integers.

(c)

$x = 0$:

$0 - 7y + 9 = 0$

$9 = 7y$

$y = \frac{9}{7}$ or $y = 1\frac{2}{7}$

Use $x = 0$ to find where the line meets the y-axis.

C is the point $(0, 1\frac{2}{7})$

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Exercise A, Question 3

Question:

The line l_1 passes through the point (9, −4) and has gradient $\frac{1}{3}$.

(a) Find an equation for l_1 in the form $ax + by + c = 0$, where a, b and c are integers.

The line l_2 passes through the origin O and has gradient −2. The lines l_1 and l_2 intersect at the point P.

(b) Calculate the coordinates of P.

Given that l_1 crosses the y-axis at the point C,

(c) calculate the exact area of ΔOCP.

Solution:

(a)

$y - (-4) = \frac{1}{3} (x - 9)$ — Use $y - y_1 = m (x - x_1)$

$y + 4 = \frac{1}{3} (x - 9)$

$y + 4 = \frac{1}{3}x - 3$ — Multiply by 3

$3y + 12 = x - 9$

$0 = x - 3y - 21$

$x - 3y - 21 = 0$ — This is the required form $ax + by + c = 0$, where a, b and c are integers.

$(a = 1 , b = -3 , c = -21)$

(b)

Equation of $l_2 : y = -2x$ — The equation of a straight line through the origin is $y = mx$.

$l_1 : \quad x - 3y - 21 = 0$

$x - 3 (-2x) - 21 = 0$ — Substitute $y = -2x$ into the equation of l_1

$x + 6x - 21 = 0$

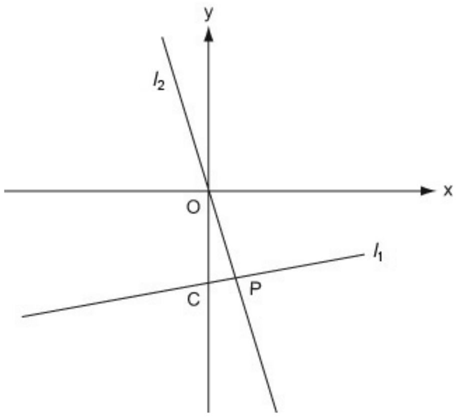
$7x = 21$

$x = 3$

$y = -2 \times 3 = -6$ — Substitute back into $y = -2x$

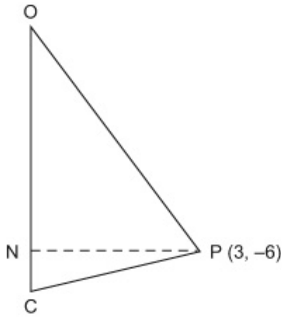
Coordinates of P : $(3 , -6)$

(c)



Use a rough sketch to show the given information

Be careful not to make any wrong assumptions. Here, for example, $\angle$ OPC is *not* 90 °



Use OC as the base and PN as the perpendicular height

Where l_1 meets the y-axis , $x = 0$.

$0 - 3y - 21 = 0$

$3y = -21$

$y = -7$

Put $x = 0$ in the equation of l_1

So OC $= 7$ and PN $= 3$

The distance of P from the y-axis is the same as its x-coordinate

Area of Δ OCP $= \frac{1}{2}$ (base × height)

$= \frac{1}{2} (7 \times 3)$

$= 10 \frac{1}{2}$

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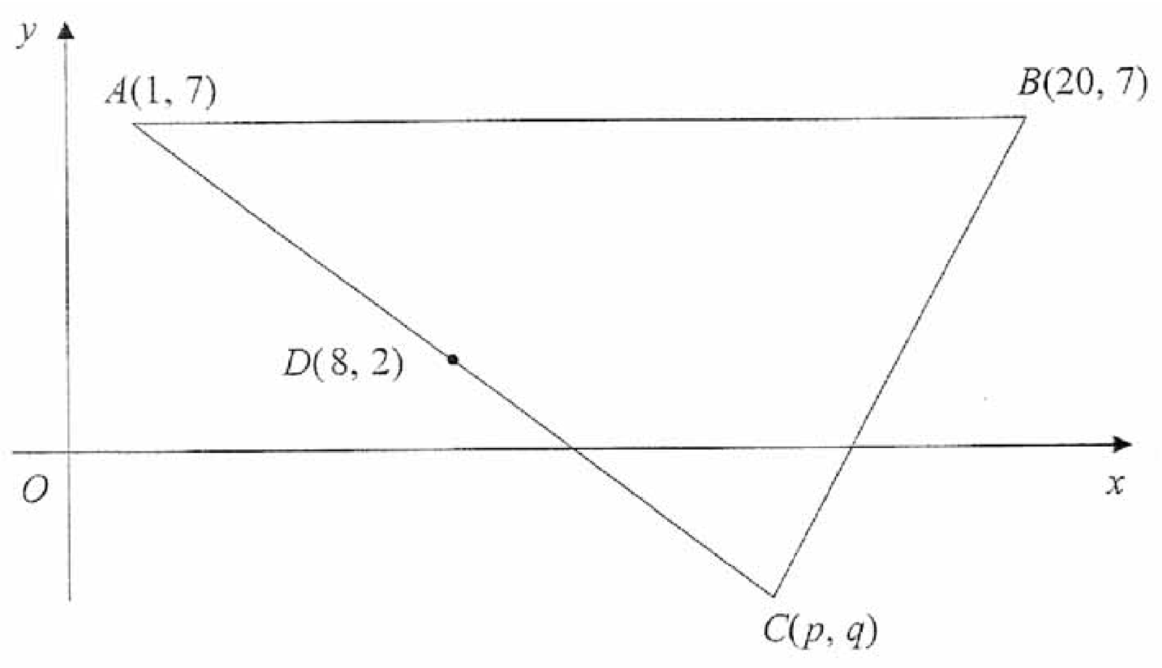
Exercise A, Question 4

Question:



The points $A(1, 7)$, $B(20, 7)$ and $C(p, q)$ form the vertices of a triangle ABC, as shown in the figure. The point $D(8, 2)$ is the mid-point of AC.

(a) Find the value of p and the value of q.

The line l, which passes through D and is perpendicular to AC, intersects AB at E.

(b) Find an equation for l, in the form $ax + by + c = 0$, where a, b and c are integers.

(c) Find the exact x-coordinate of E.

Solution:

(a)

$\left(\frac{1+p}{2} , \frac{7+q}{2} \right) = (8, 2)$ — $\left(\frac{x_1+x_2}{2} , \frac{y_1+y_2}{2} \right)$ is the mid-point of the line from (x_1, y_1) to (x_2, y_2)

$\frac{1+p}{2} = 8$ — Equate the x-coordinates

$1 + p = 16$

$p = 15$

$\frac{7+q}{2} = 2$ — Equate the y-coordinates

$7 + q = 4$

$q = -3$

(b)

Gradient of AC :

$m = \frac{2-7}{8-1} = \frac{-5}{7}$

Use the points A and D, with $m = \frac{y_2 - y_1}{x_2 - x_1}$, to find the gradient of AC (or AD) .

Gradient of l is

$-\frac{1}{\left(\frac{-5}{7}\right)} = \frac{7}{5}$

For a perpendicular line, the gradient is $-\frac{1}{m}$

$y - 2 = \frac{7}{5}(x - 8)$

The line l passes through $D(8, 2)$. So use this point in $y - y_1 = m(x - x_1)$

$y - 2 = \frac{7x}{5} - \frac{56}{5}$

$5y - 10 = 7x - 56$

$0 = 7x - 5y - 46$

Multiply by 5

$7x - 5y - 46 = 0$

$(a = 7, b = -5, c = -46)$

This is in the required form $ax + by + c = 0$, where a, b and c are integers.

(c)

The equation of AB is $y = 7$

At E :

Substitute $y = 7$ into the equation of l to find the point E.

$7x - (5 \times 7) - 46 = 0$

$7x - 35 - 46 = 0$

$7x = 81$

$x = 11\frac{4}{7}$

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Algebraic fractions
Exercise A, Question 5

Question:

The straight line l_1 has equation $y = 3x - 6$.

The straight line l_2 is perpendicular to l_1 and passes through the point (6, 2).

(a) Find an equation for l_2 in the form $y = mx + c$, where m and c are constants.

The lines l_1 and l_2 intersect at the point C.

(b) Use algebra to find the coordinates of C.

The lines l_1 and l_2 cross the x-axis at the point A and B respectively.

(c) Calculate the exact area of triangle ABC.

Solution:

(a)

The gradient of l_1 is 3. Compare with $y = mx + c$.

So the gradient of l_2 is $-\frac{1}{3}$ For a perpendicular line, the gradient is $-\frac{1}{m}$

Eqn. of l_2:

$y - 2 = -\frac{1}{3}(x - 6)$ Use $y - y_1 = m(x - x_1)$

$y - 2 = -\frac{1}{3}x + 2$

$y = -\frac{1}{3}x + 4$ This is the required form $y = mx + c$.

(b)

$y = 3x - 6$ Solve these equations simultaneously

$y = -\frac{1}{3}x + 4$

$3x - 6 = -\frac{1}{3}x + 4$

$3x + \frac{1}{3}x = 4 + 6$

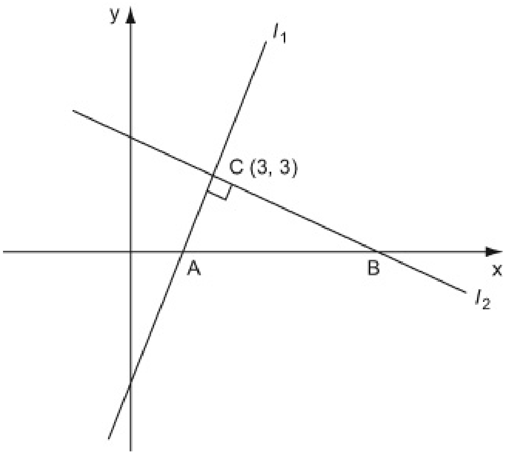
$\frac{10}{3}x = 10$ Multiply by 3 and divide by 10

$x = 3$

$y = (3 \times 3) - 6 = 3$ Substitute back into $y = 3x - 6$

The point C is $(3, 3)$

(c)



Use a rough sketch to show the given information.

Where l_1 meets the x-axis , $y = 0$:

Put $y = 0$ to find where the lines meet the x-axis

$0 = 3x - 6$

$3x = 6$

$x = 2$

A is the point (2 , 0)

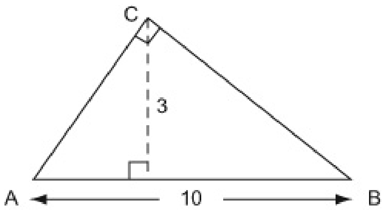
Where l_2 meets the x-axis , $y = 0$:

$0 = -\frac{1}{3}x + 4$

$\frac{1}{3}x = 4$

$x = 12$

B is the point (12 , 0)



Although ∠C is a right-angle, it is easier to use AB as the base.

AB = 10 (12 − 2)

The perpendicular height, using AB as the base, is 3

The distance of C from the x-axis is the same as its y-coordinate.

Area of Δ ABC $= \frac{1}{2}$ (base × height)

$= \frac{1}{2}$ (10 × 3)

= 15

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Algebraic fractions
Exercise A, Question 6

Question:

The line l_1 has equation $6x - 4y - 5 = 0$.

The line l_2 has equation $x + 2y - 3 = 0$.

(a) Find the coordinates of P, the point of intersection of l_1 and l_2.

The line l_1 crosses the y-axis at the point M and the line l_2 crosses the y-axis at the point N.

(b) Find the area of ΔMNP.

Solution:

(a)

$6x - 4y - 5$	$= 0 \quad (\text{i})$	Solve the equations
$x + 2y - 3$	$= 0 \quad (\text{ii})$	simultaneously
x	$= 3 - 2y$ equation (ii)	Find x in terms of y from
$6(3 - 2y) - 4y - 5$	$= 0$	Substitute into equation (i)
$18 - 12y - 4y - 5$	$= 0$	
$18 - 5$	$= 12y + 4y$	
$16y$	$= 13$	
y	$= \frac{13}{16}$	
$x = 3 - 2\left(\frac{13}{16}\right)$	$= 3 - \frac{26}{16}$	Substitute back into $x = 3 - 2y$
x	$= 1\frac{3}{8}$	

P is the point $\left(1\frac{3}{8}, \frac{13}{16}\right)$

(b)

Where l_1 meets the y-axis, $x = 0$

Put $x = 0$ to find where the lines meet the y-axis.

$0 - 4y - 5 = 0$

$4y = -5$

$y = -\frac{5}{4}$

M is the point $\left(0, \frac{-5}{4}\right)$

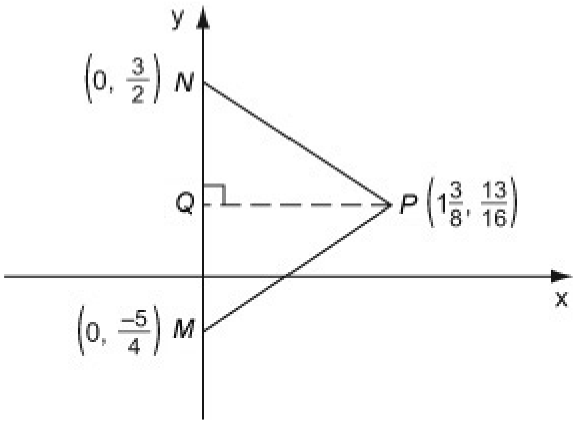
Where l_2 meets the y-axis, $x = 0$:

$0 + 2y - 3 = 0$

$2y = 3$

$y = \frac{3}{2}$

N is the point $\left(0, \frac{3}{2}\right)$



Use a rough sketch to show the information

Use MN as the base and PQ as the perpendicular height.

$MN = \frac{3}{2} + \frac{5}{4} = \frac{11}{4}$

The distance of P from the y-axis is the same as its x-coordinate

$PQ = 1\frac{3}{8} = \frac{11}{8}$

Area of $\Delta MNP = \frac{1}{2}$ (base × height)

$= \frac{1}{2}\left(\frac{11}{4} \times \frac{11}{8}\right)$

$= \frac{121}{64}$

$= 1\frac{57}{64}$

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Algebraic fractions

Exercise A, Question 7

Question:

The 5th term of an arithmetic series is 4 and the 15th term of the series is 39.

(a) Find the common difference of the series.

(b) Find the first term of the series.

(c) Find the sum of the first 15 terms of the series.

Solution:

(a)

n^{th} term $= a + (n - 1) d$

$n = 5 : \quad a + 4d = 4 \quad (\text{i})$

$n = 15 : \quad a + 14d = 39 \quad (\text{ii})$

Substitute the given values into the n^{th} term formula.

Subtract (ii)-(i)

$10d = 35$

$d = 3 \frac{1}{2}$

Solve simultaneously.

Common difference is $3 \frac{1}{2}$

(b)

$a + (4 \times 3 \frac{1}{2}) = 4$

Substitute back into equation (i).

$a + 14 = 4$

$a = - 10$

First term is −10

(c)

$S_n = \frac{1}{2}n(2a + (n-1)d)$

$n = 15$, $a = -10$, $d = 3\frac{1}{2}$ Substitute the values into the sum formula.

$S_{15} = \frac{1}{2} \times 15(-20 + (14 \times 3\frac{1}{2}))$

$= \frac{15}{2}(-20 + 49)$

$= \frac{15}{2} \times 29$

$= 217\frac{1}{2}$

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Algebraic fractions
Exercise A, Question 8

Question:

An athlete prepares for a race by completing a practice run on each of 11 consecutive days. On each day after the first day, he runs farther than he ran on the previous day. The lengths of his 11 practice runs form an arithmetic sequence with first term *a* km and common difference *d* km.

He runs 9 km on the 11th day, and he runs a total of 77 km over the 11 day period.

Find the value of *a* and the value of *d*.

Solution:

n^{th} term $= a + (n-1)d$ The distance run on the 11th day is the 11th term of the arithmetic sequence.

$n = 11: \quad a + 10d = 9$

$S_n = \frac{1}{2}n(2a + (n-1)d)$ The total distance run is the sum of the arithmetic series.

$S_n = 77,\ n = 11:$

$\frac{1}{2} \times 11(2a + 10d) = 77$

$\frac{1}{2}(2a + 10d) = 7$ It is simpler to divide each side of the equation by 11.

$a + 5d = 7$

$a + 10d = 9 \quad (i)$ Solve simultaneously

$a + 5d = 7 \quad (ii)$

Subtract (i)-(ii):

$5d = 2$

$d = \frac{2}{5}$

$a + \left(10 \times \frac{2}{5}\right) = 9$ Substitute back into equation (i).

$a + 4 = 9$

$a = 5$

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Algebraic fractions
Exercise A, Question 9

Question:

The rth term of an arithmetic series is $(2r - 5)$.

(a) Write down the first three terms of this series.

(b) State the value of the common difference.

(c) Show that $\sum_{r=1}^{n} \left(2r - 5 \right) = n \left(n - 4 \right)$.

Solution:

(a)

$r = 1: \quad 2r - 5 = -3$

$r = 2: \quad 2r - 5 = -1$

$r = 3: \quad 2r - 5 = 1$

First three terms are $-3, -1, 1$

(b)

Common difference $d = 2$

The terms increase by 2 each time $(U_{k+1} = U_{k+2})$

(c)

$\sum_{r=1}^{n} (2r - 5) = S_n$

$\sum_{r=1}^{n} (2r - 5)$ is just the sum of the series to n terms

$S_n = \frac{1}{2} n (2a + (n - 1) d)$

$a = -3, d = 2$

$S_n = \frac{1}{2} n (-6 + 2 (n - 1))$

$= \frac{1}{2} n (-6 + 2n - 2)$

$= \frac{1}{2} n (2n - 8)$

$= \frac{1}{2} n 2 (n - 4)$

$= n (n - 4)$

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Algebraic fractions
Exercise A, Question 10

Question:

Ahmed plans to save £250 in the year 2001, £300 in 2002, £350 in 2003, and so on until the year 2020. His planned savings form an arithmetic sequence with common difference £50.

(a) Find the amount he plans to save in the year 2011.

(b) Calculate his total planned savings over the 20 year period from 2001 to 2020.

Ben also plans to save money over the same 20 year period. He saves £*A* in the year 2001 and his planned yearly savings form an arithmetic sequence with common difference £60.

Given that Ben's total planned savings over the 20 year period are equal to Ahmed's total planned savings over the same period,

(c) calculate the value of *A*.

Solution:

(a)

$a = 250$ (Year 2001)

$d = 50$

Write down the values of a and d for the arithmetic series

Taking 2001 as Year 1 ($n = 1$) ,
2011 is Year 11 ($n = 11$).

Year 11 savings:

$a + (n - 1) d = 250 + (11 - 1) 50$

$= 250 + (10 \times 50)$

$= 750$

Use the term formula $a + (n - 1) d$

Year 11 savings : £ 750

(b)

$S_n = \frac{1}{2}n(2a + (n-1)d)$

Using $n = 20$,

$S_{20} = \frac{1}{2} \times 20(500 + (19 \times 50))$

$= 10(500 + 950)$

$= 10 \times 1450$

$= 14500$

The total savings will be the sum of the arithmetic series.

Total savings : £ 14 500

(c)

$a = A$ (Year2001)

$d = 60$

Write down the values of a and d for Ben's series.

$S_{20} = \frac{1}{2} \times 20(2A + (19 \times 60))$

Use the sum formula.

$S_{20} = 10(2A + 1140)$

$= 20A + 11400$

$20A + 11400 = 14500$

$20A = 14500 - 11400$

$20A = 3100$

$A = 155$

Equate Ahmed's and Ben's total savings.

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Algebraic fractions
Exercise A, Question 11

Question:

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3 ,$$
$$a_{n+1} = 3a_n - 5 , \quad n \geq 1 .$$

(a) Find the value of a_2 and the value of a_3.

(b) Calculate the value of $\sum_{r=1}^{5} a_r$.

Solution:

(a)

$a_{n+1} = 3a_n - 5$

$n = 1 : \quad a_2 = 3a_1 - 5$

$a_1 = 3 , \quad \text{so } a_2 = 9 - 5$

$a_2 = 4$

$n = 2 : a_3 = 3a_2 - 5$

$a_2 = 4 , \quad \text{so } a_3 = 12 - 5$

$a_3 = 7$

Use the given formula, with $n = 1$ and $n = 2$

(b)

$\sum_{a=1}^{5} a_r$	$= a_1 + a_2 + a_3 + a_4 + a_5$	
$n = 3 : a_4$	$= 3a_3 - 5$	This is not an arithmetic series.
$a_3 = 7$, so a_4	$= 21 - 5$	The first three terms are 3, 4, 7.
a_4	$= 16$	The differences between
$n = 4 : a_5$	$= 3a_4 - 5$	the terms are not the same.
$a_4 = 16$, so a_5	$= 48 - 5$	You cannot use a standard formula,
a_5	$= 43$	so work out each separate term and
$\sum_{a=1}^{5} a_r$	$= 3 + 4 + 7 + 16 + 43$	then add them together to find
	$= 73$	the required sum.

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Algebraic fractions
Exercise A, Question 12

Question:

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = 3a_n + 5, \quad n \geq 1,$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k.

(b) Show that $a_3 = 9k + 20$.

(c) (i) Find $\sum_{r=1}^{4} a_r$ in terms of k.

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 10.

Solution:

(a)

$$a_{n+1} = 3a_n + 5$$
$$n = 1 : a_2 = 3a_1 + 5$$
$$a_2 = 3k + 5$$

Use the given formula with $n = 1$

(b)

$$n = 2 : a_3 = 3a_2 + 5$$
$$= 3(3k + 5) + 5$$
$$= 9k + 15 + 5$$
$$a_3 = 9k + 20$$

(c)(i)

$\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

This is *not* an arithmetic series.

$n = 3 : a_4 = 3a_3 + 5$

$= 3(9k + 20) + 5$

$= 27k + 65$

You cannot use a standard formula, so work out each separate term and then add them together

$\sum_{r=1}^{4} a_r = k + (3k + 5) + (9k + 20) + (27k + 65)$

$= 40k + 90$

to find the required sum.

(ii)

$\sum_{r=1}^{4} a_r = 10(4k + 9)$

There is a factor 10, so the sum is divisible by 10.

Give a conclusion.

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Algebraic fractions
Exercise A, Question 13

Question:

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k$$
$$a_{n+1} = 2a_n - 3, \quad n \geq 1$$

(a) Show that $a_5 = 16k - 45$

Given that $a_5 = 19$, find the value of

(b) k

(c) $\sum_{r=1}^{6} a_r$

Solution:

(a)

Use the given formula with $n = 1, 2, 3$ and 4.

$$a_{n+1} = 2a_n - 3$$
$n = 1:$ $a_2 = 2a_1 - 3 = 2k - 3$

$n = 2:$ $a_3 = 2a_2 - 3 = 2(2k - 3) - 3 = 4k - 6 - 3 = 4k - 9$

$n = 3:$ $a_4 = 2a_3 - 3 = 2(4k - 9) - 3 = 8k - 18 - 3 = 8k - 21$

$n = 4:$ $a_5 = 2a_4 - 3 = 2(8k - 21) - 3 = 16k - 42 - 3$

$$a_5 = 16k - 45$$

(b)

$a_5 = 19$,

so $16k - 45 = 19$

$16k = 19 + 45$

$16k = 64$

$k = 4$

(c)

$$\sum_{r=1}^{6} a_r = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$

This is *not* an arithmetic series.

You cannot use a standard formula, so work out each separate term and then add them together to find the required sum.

$a_1 = k = 4$

$a_2 = 2k - 3 = 5$

$a_3 = 4k - 9 = 7$

$a_4 = 8k - 21 = 11$

$a_5 = 16k - 45 = 19$

From the original formula,

$a_6 = 2a_5 - 3 = (2 \times 19) - 3$

$= 35$

$$\sum_{r=1}^{6} a_r = 4 + 5 + 7 + 11 + 19 + 35$$

$= 81$

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Algebraic fractions
Exercise A, Question 14

Question:

An arithmetic sequence has first term a and common difference d.

(a) Prove that the sum of the first n terms of the series is

$$\frac{1}{2}n\left[2a+\left(n-1\right)d\right]$$

Sean repays a loan over a period of n months. His monthly repayments form an arithmetic sequence.

He repays £149 in the first month, £147 in the second month, £145 in the third month, and so on. He makes his final repayment in the nth month, where $n > 21$.

(b) Find the amount Sean repays in the 21st month.

Over the n months, he repays a total of £5000.

(c) Form an equation in n, and show that your equation may be written as

$$n^2 - 150n + 5000 = 0$$

(d) Solve the equation in part (c).

(e) State, with a reason, which of the solutions to the equation in part (c) is not a sensible solution to the repayment problem.

Solution:

(a)

S_n	$= a + (a + d) + (a + 2d) + \ldots .. + (a + (n-1)d)$	You need to know this proof . Make
Reversing the sum :	sure that you understand it, and do	
S_n	$= (a + (n-1)d) + \ldots .. + (a + 2d) + (a + d) + a$	not miss out any of the steps.
Adding these two :	When you add, each pair of terms	
$2S_n$	$= (2a + (n-1)d) + \ldots \ldots + (2a + (n-1)d)$	
$2S_n$	$= n(2a + (n-1)d)$	adds up to $2a + (n-1)d$, and there are n pairs of terms.
S_n	$= \frac{1}{2}n(2a + (n-1)d)$	

(b)

$a = 149$ (First month)

$d = -2$

Write down the values of a and d for the arithmetic series.

21st month:

$a + (n - 1)d = 149 + (20 \times -2)$
$= 149 - 40$
$= 109$

Use the term formula $a + (n - 1)d$

He repays £ 109 in the 21st month

(c)

$S_n = \frac{1}{2}n(2a + (n - 1)d)$

The total he repays will be the sum of the arithmetic series.

$= \frac{1}{2}n(298 - 2(n - 1))$

$= \frac{1}{2}n(298 - 2n + 2)$

$= \frac{1}{2}n(300 - 2n)$

$= \frac{1}{2}n2(150 - n)$

$= n(150 - n)$

$n(150 - n) = 5000$

Equate S_n to 5000

$150n - n^2 = 5000$

$n^2 - 150n + 5000 = 0$

(d)

$(n - 50)(n - 100) = 0$

$n = 50$ or $n = 100$

Always try to factorise the quadratic. The quadratic formula would be awkward here with such large numbers.

(e)

$n = 100$ is not sensible.
For example, his repayment in month 100 $(n = 100)$ would be $a + (n - 1)d$

$= 149 + (99 \times -2)$

$= 149 - 198$

$= -49$

A negative repayment is not sensible.

Check back in the context of the problem to see if the solution is sensible.

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Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions
Exercise A, Question 15

Question:

A sequence is given by

$$a_1 = 2$$
$$a_{n+1} = a_n^2 - ka_n, \quad n \geq 1,$$

where k is a constant.

(a) Show that $a_3 = 6k^2 - 20k + 16$

Given that $a_3 = 2$,

(b) find the possible values of k.

For the larger of the possible values of k, find the value of

(c) a_2

(d) a_5

(e) a_{100}

Solution:

(a)

$a_{n+1} = a_n^2 - ka_n$

$n = 1: \quad a_2 = a_1^2 - ka_1$

$= 4 - 2k$

$n = 2: \quad a_3 = a_2^2 - ka_2$

$= (4 - 2k)^2 - k(4 - 2k)$

$= 16 - 16k + 4k^2 - 4k + 2k^2$

$a_3 = 6k^2 - 20k + 16$

Use the given formula with $n = 1$ and 2.

(b)

$a_3 = 2:$

$6k^2 - 20k + 16 = 2$

$6k^2 - 20k + 14 = 0$

Divide by 2 to make solution easier

$3k^2 - 10k + 7 = 0$

$(3k - 7)(k - 1) = 0$

Try to factorise the quadratic rather than using the quadratic formula.

$k = \frac{7}{3}$ or $k = 1$

(c)

The larger k value is $\frac{7}{3}$

$a_2 = 4 - 2k = 4 - \left(2 \times \frac{7}{3}\right)$

$= 4 - \frac{14}{3} = -\frac{2}{3}$

(d)

$a_{n+1} = a_n^2 - \frac{7}{3}a_n$

$n = 3 : a_4 = a_3^2 - \frac{7}{3}a_3$

But $a_3 = 2$ is given, so

$a_4 = 2^2 - \left(\frac{7}{3} \times 2\right)$

$= 4 - \frac{14}{3} = \frac{-2}{3}$

$n = 4 : \quad a_5 = a_4^2 - \frac{7}{3}a_4$

$= \left(\frac{-2}{3}\right)^2 - \left(\frac{7}{3} \times \frac{-2}{3}\right)$

$= \frac{4}{9} + \frac{14}{9} = \frac{18}{9}$

$a_5 = 2$

Use the formula with $k = \frac{7}{3}$, for $n = 3$ and 4.

(e)

$a_2 = \frac{-2}{3}$, $a_3 = 2$

$a_4 = \frac{-2}{3}$, $a_5 = 2$

For even values of n, $a_n = \frac{-2}{3}$.

So $a_{100} = \frac{-2}{3}$

Notice that the sequence is "oscillating" between the values $\frac{-2}{3}$ and 2.

If n is even, $a_n = \frac{-2}{3}$.

If n is odd, $a_n = 2$.

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Algebraic fractions
Exercise A, Question 16

Question:

Given that

$$y = 4x^3 - 1 + 2x^{\frac{1}{2}} , \quad x > 0 ,$$

find $\frac{dy}{dx}$.

Solution:

$y = 4x^3 - 1 + 2x^{\frac{1}{2}}$

For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

$\frac{dy}{dx} = (4 \times 3x^2) + (2 \times \frac{1}{2}x^{-\frac{1}{2}})$

Differentiating the constant -1 gives zero.

$\frac{dy}{dx} = 12x^2 + x^{-\frac{1}{2}}$

It is better to write down an unsimplified version of the answer first (in case you make a mistake when simplifying).

(Or: $\frac{dy}{dx} = 12x^2 + \frac{1}{x^{\frac{1}{2}}}$

Or: $\frac{dy}{dx} = 12x^2 + \frac{1}{\sqrt{x}}$)

It is not necessary to change your answer into one of these forms.

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Algebraic fractions
Exercise A, Question 17

Question:

Given that $y = 2x^2 - \frac{6}{x^3}$, $x \neq 0$,

(a) find $\frac{dy}{dx}$,

(b) find $\int y \ dx$.

Solution:

(a)

$y = 2x^2 - \frac{6}{x^3}$

$= 2x^2 - 6x^{-3}$

Use $\frac{1}{x^n} = x^{-n}$

$\frac{dy}{dx} = (2 \times 2x^1) - (6 \times -3x^{-4})$

For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

$\frac{dy}{dx} = 4x + 18x^{-4}$

Write down an unsimplified version of the answer first.

(Or: $\frac{dy}{dx} = 4x + \frac{18}{x^4}$)

It is not necessary to change your answer into this form.

(b)

$\int (2x^2 - 6x^{-3})\,dx$

$= \frac{2x^3}{3} - \frac{6x^{-2}}{-2} + C$

$= \frac{2x^3}{3} + 3x^{-2} + C$

(Or: $\frac{2x^3}{3} + \frac{3}{x^2} + C$)

Use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

Do not forget to include the constant of integration, C.

Write down an unsimplified version of the answer first

It is not necessary to change your answer into this form.

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Algebraic fractions
Exercise A, Question 18

Question:

Given that $y = 3x^2 + 4\sqrt{x}$, $x > 0$, find

(a) $\frac{dy}{dx}$,

(b) $\frac{d^2y}{dx^2}$,

(c) $\int y \, dx$.

Solution:

(a)

$y = 3x^2 + 4\sqrt{x}$ Use $\sqrt{x} = x^{\frac{1}{2}}$

$= 3x^2 + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = (3 \times 2x^1) + (4 \times \frac{1}{2}x^{-\frac{1}{2}})$ For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

$\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$ Write down an unsimplified version of the answer first.

(Or: $\frac{dy}{dx} = 6x + \frac{2}{x^{\frac{1}{2}}}$

Or: $\frac{dy}{dx} = 6x + \frac{2}{\sqrt{x}}$) It is not necessary to change your answer into one of these forms

(b)

$\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$

Differentiate again

$\frac{d^2y}{dx^2} = 6 + \left(2 \times \frac{-1}{2} x^{-\frac{3}{2}}\right)$

$= 6 - x^{-\frac{3}{2}}$

(Or: $\frac{d^2y}{dx^2} = 6 - \frac{1}{x^{\frac{3}{2}}}$ Or: $\frac{d^2y}{dx^2} = 6 - \frac{1}{x\sqrt{x}}$)

It is not necessary to change your answer into one of these forms.

$x^{\frac{3}{2}} = x^1 \times x^{\frac{1}{2}} = x\sqrt{x}$

(c)

$\int \left(3x^2 + 4x^{\frac{1}{2}}\right) dx$

Use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ Do not forget to include the constant of integration, C

$= \frac{3x^3}{3} + \frac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$

Write down an unsimplified version of the answer first.

$= x^3 + 4\left(\frac{2}{3}\right) x^{\frac{3}{2}} + C$

$= x^3 + \frac{8}{3} x^{\frac{3}{2}} + C$

(Or: $x^3 + \frac{8}{3} x\sqrt{x} + C$)

It is not necessary to change your answer into this form.

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Algebraic fractions
Exercise A, Question 19

Question:

(i) Given that $y = 5x^3 + 7x + 3$, find

(a) $\frac{dy}{dx}$,

(b) $\frac{d^2y}{dx^2}$.

(ii) Find $\int \left(1 + 3\sqrt{x} - \frac{1}{x^2} \right) dx$.

Solution:

(i)

$y = 5x^3 + 7x + 3$

(a)

$\frac{dy}{dx} = (5 \times 3x^2) + (7 \times 1x^0)$

For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$.

Differentiating the constant 3 gives zero.

$\frac{dy}{dx} = 15x^2 + 7$

Use $x^0 = 1$

Differentiating Kx gives K .

(b)

$\frac{dy}{dx} = 15x^2 + 7$

Differentiate again

$\frac{d^2y}{dx^2} = (15 \times 2x^1)$

$= 30x$

(ii)

$\int (1 + 3\sqrt{x} - \frac{1}{x^2}) \, dx$

Use $\sqrt{x} = x^{\frac{1}{2}}$ and $\frac{1}{x^n} = x^{-n}$

$= \int (1 + 3x^{\frac{1}{2}} - x^{-2}) \, dx$

Use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

Do not forget to include the constant of integration C.

$= x + \frac{3x^{\frac{3}{2}}}{(\frac{3}{2})} - \frac{x^{-1}}{(-1)} + C$

$= x + (3 \times \frac{2}{3} x^{\frac{3}{2}}) + x^{-1} + C$

$= x + 2x^{\frac{3}{2}} + x^{-1} + C$

(Or: $x + 2x\sqrt{x} + \frac{1}{x} + C$)

It is not necessary to change your answer into this form.

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Algebraic fractions
Exercise A, Question 20

Question:

The curve C has equation $y = 4x + 3x^{\frac{3}{2}} - 2x^2$, $x > 0$.

(a) Find an expression for $\frac{dy}{dx}$.

(b) Show that the point $P(4, 8)$ lies on C.

(c) Show that an equation of the normal to C at the point P is

$$3y = x + 20 .$$

The normal to C at P cuts the x-axis at the point Q.

(d) Find the length PQ, giving your answer in a simplified surd form.

Solution:

(a)

$y = 4x + 3x^{\frac{3}{2}} - 2x^2$

$\frac{dy}{dx} = (4 \times 1x^0) + (3 \times \frac{3}{2}x^{\frac{1}{2}}) - (2 \times 2x^1)$

For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

$\frac{dy}{dx} = 4 + \frac{9}{2}x^{\frac{1}{2}} - 4x$

(b)

For $x = 4$,

$y = (4 \times 4) + (3 \times 4^{\frac{3}{2}}) - (2 \times 4^2)$

$x^{\frac{3}{2}} = x^1 \times x^{\frac{1}{2}} = x\sqrt{x}$

$= 16 + (3 \times 4 \times 2) - 32$

$= 16 + 24 - 32 = 8$

So P (4 , 8) lies on C

(c)

For $x = 4$, $\frac{dy}{dx}$

$$\frac{dy}{dx} = 4 + \left(\frac{9}{2} \times 4^{\frac{1}{2}}\right) - (4 \times 4)$$

$$= 4 + \left(\frac{9}{2} \times 2\right) - 16$$

$$= 4 + 9 - 16 = -3$$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

The gradient of the normal at P is $\frac{1}{3}$

The normal is perpendicular to the tangent, so the gradient is $-\frac{1}{m}$

Equation of the normal :

$$y - 8 = \frac{1}{3}(x - 4)$$

Use $y - y_1 = m(x - x_1)$

$$y - 8 = \frac{x}{3} - \frac{4}{3}$$

Multiply by 3

$$3y - 24 = x - 4$$

$$3y = x + 20$$

(d)

$y = 0$: $0 = x + 20$

$x = -20$

Use $y = 0$ to find where the normal cuts the x-axis.

Q is the point $(-20, 0)$

$$PQ = \sqrt{(4 - -20)^2 + (8 - 0)^2}$$

$$= \sqrt{24^2 + 8^2}$$

$$= \sqrt{576 + 64}$$

$$= \sqrt{640}$$

$$= \sqrt{64} \times \sqrt{10}$$

$$= 8\sqrt{10}$$

The distance between two points is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

To simplify the surd, find a factor which is an exact square (here $64 = 8^2$)

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Algebraic fractions
Exercise A, Question 21

Question:

The curve C has equation $y = 4x^2 + \frac{5-x}{x}$, $x \neq 0$. The point P on C has x-coordinate 1.

(a) Show that the value of $\frac{dy}{dx}$ at P is 3.

(b) Find an equation of the tangent to C at P.

This tangent meets the x-axis at the point $(k, 0)$.

(c) Find the value of k.

Solution:

(a)

$y = 4x^2 + \frac{5-x}{x}$ — Divide $5 - x$ by x

$= 4x^2 + 5x^{-1} - 1$ — For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

$\frac{dy}{dx} = (4 \times 2x^1) + (5x - 1x^{-2})$ — Differentiating the constant -1 gives zero

$\frac{dy}{dx} = 8x - 5x^{-2}$

At P, $x = 1$, so

$\frac{dy}{dx} = (8 \times 1) - (5 \times 1^{-2})$ — $1^{-2} = \frac{1}{1^2} = \frac{1}{1} = 1$

$= 8 - 5 = 3$

(b)

At $x = 1$, $\frac{dy}{dx} = 3$ — The value of $\frac{dy}{dx}$ is the gradient of the tangent

At $x = 1$, $y = (4 \times 1^2) + \frac{5-1}{1}$

$y = 4 + 4 = 8$

Equation of the tangent :

$y - 8 = 3(x - 1)$ — Use $y - y_1 = m(x - x_1)$

$y = 3x + 5$

(c)

$y = 0$: $0 = 3x + 5$ — Use $y = 0$ to find where the tangent meets the x-axis

$3x = -5$

$x = -\frac{5}{3}$

So K $= -\frac{5}{3}$

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Algebraic fractions
Exercise A, Question 22

Question:

The curve C has equation $y = \frac{1}{3}x^3 - 4x^2 + 8x + 3$.

The point P has coordinates (3, 0).

(a) Show that P lies on C.

(b) Find the equation of the tangent to C at P, giving your answer in the form $y = mx + c$, where m and c are constants.

Another point Q also lies on C. The tangent to C at Q is parallel to the tangent to C at P.

(c) Find the coordinates of Q.

Solution:

(a)

$y = \frac{1}{3}x^3 - 4x^2 + 8x + 3$

At $x = 3$,

$y = \left(\frac{1}{3} \times 3^3 \right) - \left(4 \times 3^2 \right) + \left(8 \times 3 \right) + 3$

$= 9 - 36 + 24 + 3$

$= 0$

So P (3 , 0) lies on C

(b)

$\frac{dy}{dx} = \left(\frac{1}{3} \times 3x^2 \right) - \left(4 \times 2x^1 \right) + \left(8 \times 1x^0 \right)$

For $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$

Differentiating the constant 3 gives zero.

$= x^2 - 8x + 8$

At $x = 3$,

$\frac{dy}{dx} = 3^2 - \left(8 \times 3 \right) + 8$

$= 9 - 24 + 8 = -7$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

Equation of the tangent :

$y - 0 = -7 \left(x - 3 \right)$

Use $y - y_1 = m \left(x - x_1 \right)$

$y = -7x + 21$

This is in the required form $y = mx + c$

(c)

At Q, $\frac{dy}{dx} = -7$

If the tangents are parallel, they have the same gradient.

$x^2 - 8x + 8 = -7$

$x^2 - 8x + 15 = 0$

$\left(x - 3 \right) \left(x - 5 \right) = 0$

$x = 3$ or $x = 5$ $\quad$ $x = 3$ at the point P

For Q, $x = 5$

$y = \left(\frac{1}{3} \times 5^3 \right) - \left(4 \times 5^2 \right) + \left(8 \times 5 \right) + 3$

Substitute $x = 5$ back into the equation of C

$= \frac{125}{3} - 100 + 40 + 3$

$= -15 \frac{1}{3}$

Q is the point $\left(5 , -15 \frac{1}{3} \right)$

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Algebraic fractions
Exercise A, Question 23

Question:

$$f(x) = \frac{(2x+1)(x+4)}{\sqrt{x}}, \quad x > 0$$

(a) Show that f(x) can be written in the form $Px^{\frac{3}{2}} + Qx^{\frac{1}{2}} + Rx^{-\frac{1}{2}}$, stating the values of the constants P, Q and R.

(b) Find $f'(x)$.

(c) Show that the tangent to the curve with equation $y = f(x)$ at the point where $x = 1$ is parallel to the line with equation $2y = 11x + 3$.

Solution:

(a)

$$f(x) = \frac{(2x+1)(x+4)}{\sqrt{x}}$$

$$= \frac{2x^2 + 9x + 4}{\sqrt{x}}$$

Divide each term by $x^{\frac{1}{2}}$, remembering that $x^m \div x^n = x^{m-n}$

$$= 2x^{\frac{3}{2}} + 9x^{\frac{1}{2}} + 4x^{\frac{-1}{2}} .$$

$P = 2$, $Q = 9$, $R = 4$

(b)

$$f'(x) = \left(2 \times \frac{3}{2}x^{\frac{1}{2}}\right) + \left(9 \times \frac{1}{2}x^{\frac{-1}{2}}\right) + \left(4 \times \frac{-1}{2}x^{\frac{-3}{2}}\right)$$

$f'(x)$ is the derivative of $f(x)$, so differentiate

$$f'(x) = 3x^{\frac{1}{2}} + \frac{9}{2}x^{\frac{-1}{2}} - 2x^{\frac{-3}{2}}$$

(c)

At $x = 1$,

$f'(1) = \left(3 \times 1^{\frac{1}{2}}\right) + \left(\frac{9}{2} \times 1^{\frac{-1}{2}}\right) - \left(2 \times 1^{\frac{-3}{2}}\right)$ $f'(1)$ is the gradient of the tangent at $x = 1$

$= 3 + \frac{9}{2} - 2 = \frac{11}{2}$ $1^n = 1$ for any n.

The line $2y = 11x + 3$ is

$y = \frac{11}{2}x + \frac{3}{2}$ Compare with $y = mx + c$

The gradient is $\frac{11}{2}$

So the tangent to the curve where $x = 1$ is parallel to this line, since the gradients are equal. Give a conclusion, with a reason.

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Algebraic fractions
Exercise A, Question 24

Question:

The curve C with equation $y = f(x)$ passes through the point (3, 5).

Given that $f'(x) = x^2 + 4x - 3$, find f(x).

Solution:

$f'(x) = x^2 + 4x - 3$

To find $f(x)$ from $f'(x)$, integrate.

$f(x) = \frac{x^3}{3} + \frac{4x^2}{2} - 3x + C$

Use $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$= \frac{x^3}{3} + 2x^2 - 3x + C$

Do not forget to include the constant of integration C.

When $x = 3$, $f(x) = 5$, so

$\frac{3^3}{3} + (2 \times 3^2) - (3 \times 3) + C = 5$

The curve passes through $(3, 5)$, so $f(3) = 5$.

$9 + 18 - 9 + C = 5$

$C = -13$

$f(x) = \frac{x^3}{3} + 2x^2 - 3x - 13$

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Algebraic fractions
Exercise A, Question 25

Question:

The curve with equation $y = f(x)$ passes through the point (1, 6). Given that

$$f'(x) = 3 + \frac{5x^2 + 2}{x^{\frac{1}{2}}}, \ x > 0,$$

find f(x) and simplify your answer.

Solution:

$f'(x) = 3 + \dfrac{5x^2 + 2}{x^{\frac{1}{2}}}$

Divide $5x^2 + 2$ by $x^{\frac{1}{2}}$, remembering that $x^m \div x^n = x^{m-n}$

$= 3 + 5x^{\frac{3}{2}} + 2x^{-\frac{1}{2}}$

To find $f(x)$ from $f'(x)$, integrate.

$f(x) = 3x + \dfrac{5x^{\frac{5}{2}}}{\left(\frac{5}{2}\right)} + \dfrac{2x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$

Use $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$.

$= 3x + \left(5 \times \frac{2}{5}x^{\frac{5}{2}}\right) + \left(2 \times \frac{2}{1}x^{\frac{1}{2}}\right) + C$

Do not forget to include the constant of integration C.

$= 3x + 2x^{\frac{5}{2}} + 4x^{\frac{1}{2}} + C$

When $x = 1$, $f(x) = 6$, so

The curve passes through $(1, 6)$, so $f(1) = 6$

$(3 \times 1) + (2 \times 1^{\frac{5}{2}}) + (4 \times 1^{\frac{1}{2}}) + C = 6$

$3 + 2 + 4 + C = 6$

$1^n = 1$ for any n.

$C = -3$

$f(x) = 3x + 2x^{\frac{5}{2}} + 4x^{\frac{1}{2}} - 3$

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Algebraic fractions
Exercise A, Question 26

Question:

For the curve C with equation $y = f(x)$,

$$\frac{dy}{dx} = x^3 + 2x - 7$$

(a) Find $\frac{d^2y}{dx^2}$

(b) Show that $\frac{d^2y}{dx^2} \geq 2$ for all values of x.

Given that the point $P(2, 4)$ lies on C,

(c) find y in terms of x,

(d) find an equation for the normal to C at P in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

(a)

$\frac{dy}{dx} = x^3 + 2x - 7$

$\frac{d^2y}{dx^2} = 3x^2 + 2$

Differentiate to find the second derivative

(b)

$x^2 \geq 0$ for any (real) x.

So $3x^2 \geq 0$

So $3x^2 + 2 \geq 2$

So $\frac{d^2y}{dx^2} \geq 2$ for all values of x.

The square of a real number cannot be negative.

Give a conclusion.

(c)

$\frac{dy}{dx} = x^3 + 2x - 7$

Integrate $\frac{dy}{dx}$ to find y in terms of x.

$y = \frac{x^4}{4} + \frac{2x^2}{2} - 7x + C$

$= \frac{x^4}{4} + x^2 - 7x + C$

Do not forget to include the constant of integration C.

When $x = 2$, $y = 4$, so

$4 = \frac{2^4}{4} + 2^2 - (7 \times 2) + C$

$4 = 4 + 4 - 14 + C$

$C = +10$

$y = \frac{x^4}{4} + x^2 + 7x + 10$

Use the fact that $P(2, 4)$ lies on the curve.

(d)

For $x = 2$,

$\frac{dy}{dx} = 2^3 + (2 \times 2) - 7$

$= 8 + 4 - 7 = 5$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

The gradient of the normal at P is $\frac{-1}{5}$

The normal is perpendicular to the tangent, so the gradient is $-\frac{1}{m}$

Equation of the normal:

$y - 4 = \frac{-1}{5}(x - 2)$

Use $y - y_1 = m(x - x_1)$

$y - 4 = \frac{-x}{5} + \frac{2}{5}$

Multiply by 5

$5y - 20 = -x + 2$

$x + 5y - 22 = 0$

This is in the required form $ax + by + c = 0$, where a, b and c are integers.

Solutionbank C1

Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions

Exercise A, Question 27

Question:

For the curve C with equation $y = f(x)$,

$$\frac{dy}{dx} = \frac{1 - x^2}{x^4}$$

Given that C passes through the point $\left(\frac{1}{2}, \frac{2}{3} \right)$,

(a) find y in terms of x.

(b) find the coordinates of the point on C at which $\frac{dy}{dx} = 0$.

Solution:

(a)

$\frac{dy}{dx} = \frac{1 - x^2}{x^4}$ Divide $1 - x^2$ by x^4

$= x^{-4} - x^{-2}$

$y = \frac{x^{-3}}{-3} - \frac{x^{-1}}{-1} + C$ Integrate $\frac{dy}{dx}$ to find y in terms of x. Do not forget to include the constant of integration C.

$= \frac{-x^{-3}}{3} + x^{-1} + C$

$y = \frac{-1}{3x^3} + \frac{1}{x} + C$ Use $x^{-n} = \frac{1}{x^n}$. This will make it easier to calculate values at the next stage.

When $x = \frac{1}{2}$, $y = \frac{2}{3}$, so

$\frac{2}{3} = -\frac{8}{3} + 2 + C$ Use the fact that $\left(\frac{1}{2}, \frac{2}{3} \right)$ lies on the curve.

$C = \frac{2}{3} + \frac{8}{3} - 2 = \frac{4}{3}$

$y = \frac{-1}{3x^3} + \frac{1}{x} + \frac{4}{3}$

(b)

$\frac{1 - x^2}{x^4} = 0$ If a fraction is equal to zero, its numerator must be zero.

$1 - x^2 = 0$

$x^2 = 1$

$x = 1$ or $x = -1$

$x = 1: \quad y = \frac{-1}{3} + 1 + \frac{4}{3}$

$y = 2$

$x = -1: \quad y = \frac{1}{3} - 1 + \frac{4}{3}$

$y = \frac{2}{3}$

The points are $(1, 2)$ and $\left(-1, \frac{2}{3}\right)$

Solutionbank C1
Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions
Exercise A, Question 28

Question:

The curve C with equation $y = f(x)$ passes through the point (5, 65).

Given that $f'(x) = 6x^2 - 10x - 12$,

(a) use integration to find f (x).

(b) Hence show that $f(x) = x(2x+3)(x-4)$.

(c) Sketch C, showing the coordinates of the points where C crosses the x-axis.

Solution:

(a)

$f'(x) = 6x^2 - 10x - 12$

To find $f(x)$ from $f'(x)$, integrate

$f(x) = \frac{6x^3}{3} - \frac{10x^2}{2} - 12x + C$

Do not forget to include the constant of integration C.

When $x = 5$, $y = 65$, so

$65 = \frac{6 \times 125}{3} - \frac{10 \times 25}{2} - 60 + C$

Use the fact that the curve passes through $(5, 65)$

$65 = 250 - 125 - 60 + C$

$C = 65 + 125 + 60 - 250$

$C = 0$

$f(x) = 2x^3 - 5x^2 - 12x$

(b)

$f(x) = x(2x^2 - 5x - 12)$

$f(x) = x(2x+3)(x-4)$

(c)

Curve meets x-axis where $y = 0$

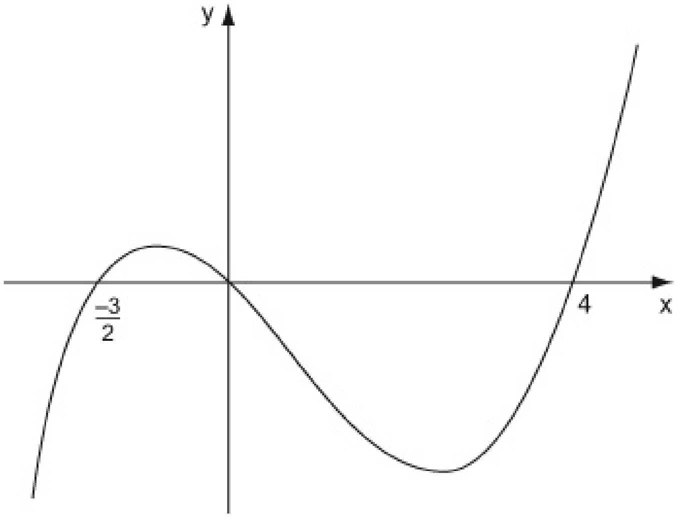
$x(2x+3)(x-4) = 0$ Put $y = 0$ and solve for x

$x = 0, x = -\frac{3}{2}, x = 4$

When $x \to \infty$, $y \to \infty$

When $x \to -\infty$, $y \to -\infty$

Check what happens to y for large positive and negative values of x.



Crosses x-axis at $\left(\frac{-3}{2}, 0\right)$, $(0, 0)$, $(4, 0)$

Solutionbank C1
Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions
Exercise A, Question 29

Question:

The curve C has equation $y = x^2 \left(x - 6 \right) + \frac{4}{x}$, $x > 0$.

The points P and Q lie on C and have x-coordinates 1 and 2 respectively.

(a) Show that the length of PQ is $\sqrt{170}$.

(b) Show that the tangents to C at P and Q are parallel.

(c) Find an equation for the normal to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

Solution:

(a)

$y = x^2 (x - 6) + \frac{4}{x}$

At P, $x = 1$,

$y = 1 (1 - 6) + \frac{4}{1} = - 1$

P is $(1 , - 1)$

At Q, $x = 2$,

$y = 4 (2 - 6) + \frac{4}{2} = - 14$

Q is $(2 , - 14)$

$PQ = \sqrt{ (2 - 1) ^2 + (- 14 - (- 1)) ^2 }$

$= \sqrt{ (1^2 + (- 13) ^2) }$

$= \sqrt{ (1 + 169) } = \sqrt{170}$

The distance between two points is $\sqrt{ (x_2 - x_1) ^2 + (y_2 - y_1) ^2 }$

(b)

$y = x^3 - 6x^2 + 4x^{ - 1}$

$\frac{dy}{dx} = 3x^2 - (6 \times 2x') + (4x - 1x^{ - 2})$

$= 3x^2 - 12x - 4x^{ - 2}$

At $x = 1$,

$\frac{dy}{dx} = 3 - 12 - 4 = - 13$

The value of $\frac{dy}{dx}$ is the gradient of the tangent.

At $x = 2$,

$\frac{dy}{dx} = (3 \times 4) - (12 \times 2) - (4 \times 2^{ - 2})$

$= 12 - 24 - \frac{4}{4} = - 13$

At P and also at Q the gradient is −13, so the tangents are parallel (equal gradients).

Give a conclusion

(c)

The gradient of the normal at P is $-\frac{1}{-13} = \frac{1}{13}$

The normal is perpendicular to the tangent, so the gradient is $-\frac{1}{m}$

Equation of the normal:

$y-(-1) = \frac{1}{13}(x-1)$ Use $y - y_1 = m(x - x_1)$

$y+1 = \frac{x}{13} - \frac{1}{13}$ Multiply by 13

$13y + 13 = x - 1$

$x - 13y - 14 = 0$ This is in the required form $ax + by + c = 0$, where a, b and c are integers.

Solutionbank C1
Edexcel Modular Mathematics for AS and A-Level

Algebraic fractions
Exercise A, Question 30

Question:

(a) Factorise completely $x^3 - 7x^2 + 12x$.

(b) Sketch the graph of $y = x^3 - 7x^2 + 12x$, showing the coordinates of the points at which the graph crosses the x-axis.

The graph of $y = x^3 - 7x^2 + 12x$ crosses the positive x-axis at the points A and B.

The tangents to the graph at A and B meet at the point P.

(c) Find the coordinates of P.

Solution:

(a)

$x^3 - 7x^2 + 12x$ — x is a common factor

$= x (x^2 - 7x + 12)$

$= x (x - 3) (x - 4)$

(b)

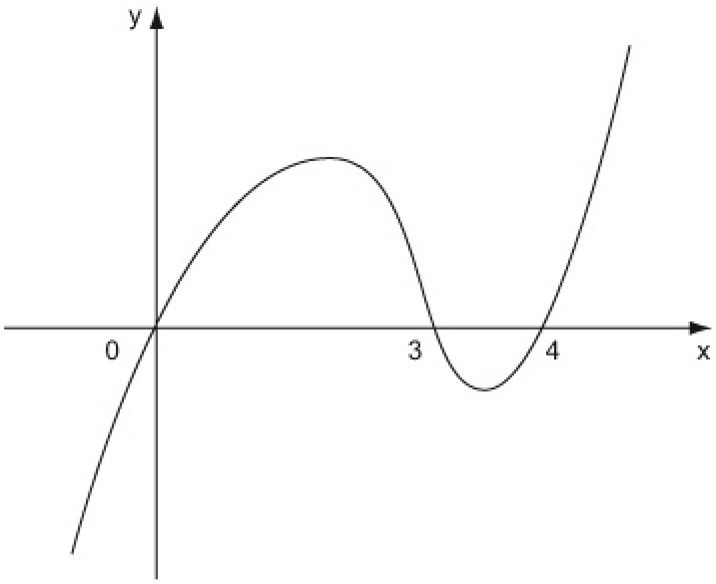
Curve meets x-axis where $y = 0$.

$x (x - 3) (x - 4) = 0$ — Put $y = 0$ and solve for x .

$x = 0 , x = 3 , x = 4$

When $x \to \infty , y \to \infty$ — Check what happens to y for large positive and negative values of x

When $x \to - \infty , y \to - \infty$



Crosses x-axis at $(0 , 0)$, $(3 , 0)$, $(4 , 0)$

(c)

A and B are $(3, 0)$ and $(4, 0)$

$\frac{dy}{dx} = 3x^2 - 14x + 12$

At $x = 3$, (A) — The value of $\frac{dy}{dx}$ is the gradient of the tangent.

$\frac{dy}{dx} = 27 - 42 + 12 = -3$

At $x = 4$ (B)

$\frac{dy}{dx} = 48 - 56 + 12 = 4$

Tangent at A: — Use $y - y_1 = m(x - x_1)$

$y - 0 = -3(x - 3)$

$y = -3x + 9$ (i)

Tangent at B:

$y - 0 = 4(x - 4)$

$y = 4x - 16$ (ii)

Subtract (ii) − (i):

$0 = 7x - 25$ — Solve (i) and (ii) simultaneously to find the intersection point of the tangents

$x = \frac{25}{7}$

Substituting back into (i):

$y = -\frac{75}{7} + 9 = -\frac{12}{7}$

P is the point $\left(\frac{25}{7}, \frac{-12}{7}\right)$