Math 6345 - AODE's

Putzer's Algorithm (3x3)

$$e^{At} = P_0 r_1 + P_1 r_2 + P_2 r_3$$

where $P_0 = I$, $P_1 = A - \lambda I$, $P_2 = (A-\lambda I)(A-\lambda_2 I)$

& $\dot{r}_1 = \lambda r_1$, $\dot{r}_2 = \lambda_2 r_2 + r_1$, $\dot{r}_3 = \lambda_3 r_3 + r_2$

with $r_1(0) = 1$, $r_2(0) = 0$, $r_3(0) = 0$

Ex!
$$\dot{\bar{x}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ -2 & -2 & -1 \end{pmatrix} \bar{x}$$

Now
$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ -2 & -2 & -1-\lambda \end{vmatrix} = (\lambda-1)^3 = 0$$

so eigenvalue $\lambda = 1, 1, 1$

we first find e^{At} via the eigenvalue/vector way, so when $\lambda = 1$

$$(A - I)\bar{a} = 0 \quad \text{so} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad c_1 + c_2 + c_3 = 0 \quad a \quad c_3 = -c_1 - c_2$$

$$\text{so} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} c_2$$

so there are 2 linearly independent eigenvectors

$$\text{so} \quad \bar{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^t, \quad \bar{x}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^t$$

For the third sol^n, we seek it in the form

$$\bar{x} = \bar{u} t e^{\lambda t} + \bar{v} e^{\lambda t}$$

sub into system gives

$$\dot{x} = \bar{u} e^{\lambda t} + \lambda \bar{u} t e^{\lambda t} + \lambda \bar{v} e^{\lambda t}$$

so $\dot{x} = A\bar{x} \Rightarrow \bar{u} e^{\lambda t} + \lambda \bar{u} t e^{\lambda t} + \lambda \bar{v} e^{\lambda t} = A\bar{u} t e^{\lambda t} + A\bar{v} e^{\lambda t}$

and comparing terms $(\ )t + (\ ) = 0$

$$\Rightarrow \quad \lambda \bar{u} = A\bar{u} \quad \text{or} \quad (A - \lambda I)\bar{u} = \bar{0}$$

$$\bar{u} + \lambda \bar{v} = A\bar{v} \qquad (A - \lambda I)\bar{v} = \bar{u}$$

$\because$ $(A - I)$ is already given then we want to solve

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad (*)$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Since we already know 2 sol^n's $(*)$, will they work?

d. will $\bar{u} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ work

so second soln

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} v_1 + v_2 + v_3 = 1 \\ -2v_1 - 2v_2 - 2v_3 = -1 \end{matrix}$$

and there is no sol^n to this! So we need a combination of

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \; \& \; \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

so

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

so

$$\left. \begin{matrix} v_1 + v_2 + v_3 = a \\ v_1 + v_2 + v_3 = b \end{matrix} \right\} \Rightarrow a = b$$

$$-2v_1 - 2v_2 - 2v_3 = -a - b \Rightarrow -2v_1 - 2v_2 - 2v_3 = -2a$$

$$v_1 + v_2 + v_3 = a \; ✓$$

So pick $a = b = 1$

so $\bar{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ & $v_1 + v_2 + v_3 = 1$

so pick $\bar{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

so 3^{rd} sol^n is

$$\bar{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} te^t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t$$

the fundament matrix is

$$\Phi = \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & te^t \\ -e^t & -e^t & (1-2t)e^t \end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Now we need the inverse!

so we use some linear algebra

$$(A \mid I) \to (I \mid A^{-1})$$

so
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$R_1 + R_2 + R_3 \to R_3$$

so $e^{At} = \Phi(t)\Phi^{-1}(0)$

$$= \begin{pmatrix} e^t & 0 & te^t \\ 0 & e^t & te^t \\ -e^t & -e^t & (1-2t)e^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^t + te^t & te^t & te^t \\ te^t & (1+t)e^t & te^t \\ -2te^t & -2te^t & (1-2t)e^t \end{pmatrix}$$

Note $e^0 = I$ ✓

Now Putzer's Algorithm

so $P_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad P_1 = A - I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}$

$$P_2 = (A-I)(A-I) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} = (0)$$

$\dot{r}_1 = r_1$ so $r_1 = c_1 e^t$ $\quad r_1(0) = 1 \Rightarrow c_1 = 1$ so $r_1 = e^t$

$\dot{r}_2 = r_2 + e^t$ $\quad \dot{r}_2 - r_2 = e^t$ $\quad \mu = e^{-t}$ so $\frac{d}{dt}(e^{-t} r_2) = 1$

$e^{-t} r_2 = t + c_2$ $\quad r_2(0) = 0 \Rightarrow c_2 = 0$ so $r_2 = t e^t$

$\dot{r}_3 = r_3 + t e^t$ (not really needed b/c $P_2 = 0$)

but μ is still the same so

$$\frac{d}{dt} e^{-t} r_3 = t \Rightarrow e^{-t} r_3 = t^2/2 + c_3$$

$$r_3(0) = 0 \Rightarrow c_3 = 0 \text{ so } r_3 = \frac{t^2}{2} e^t$$

$$e^{At} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^t + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{pmatrix} te^t$$

$$= \begin{pmatrix} e^t + te^t & te^t & te^t \\ te^t & (1+t)e^t & te^t \\ -2te^t & -2te^t & (1-2t)e^t \end{pmatrix}$$

as this we saw using the eigenvalue - vector method.