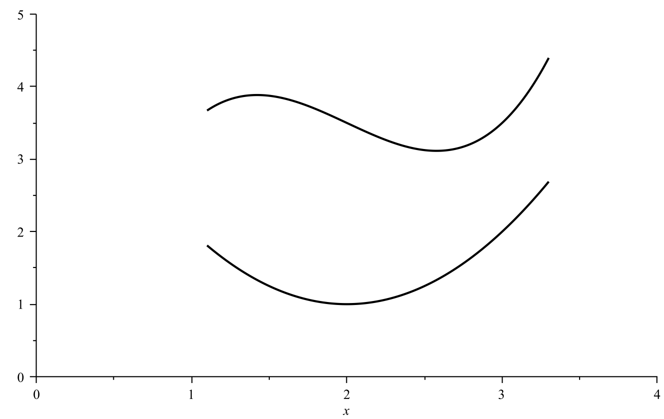
Calculus 3 - Triple Integrals

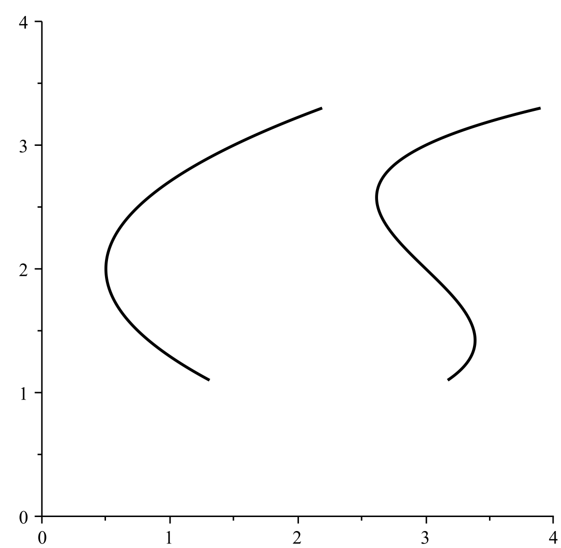
So far we have introduced double integrals

$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx \tag{1}$$



or

$$\int_c^d \int_{G(y)}^{H(y)} f(x,y)dxdy \tag{2}$$



Now we move to triple integrals.

Single integral is over a interval

$$\int_I f(x)dx \tag{3}$$

Double integral is over a plane region

$$\iint_R f(x,y)dA \tag{4}$$

Triple integral is over a volume

$$\iiint_V f(x,y,z)dV \tag{5}$$

A single integral - point to point

$$\int_a^b f(x)dx \tag{6}$$

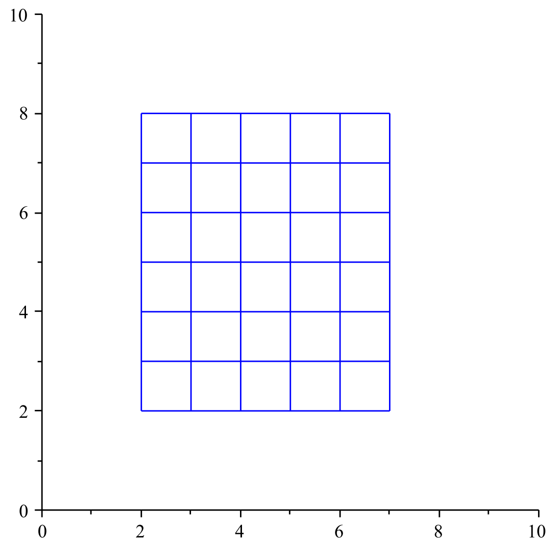
a double integral - curve to curve, then point to point

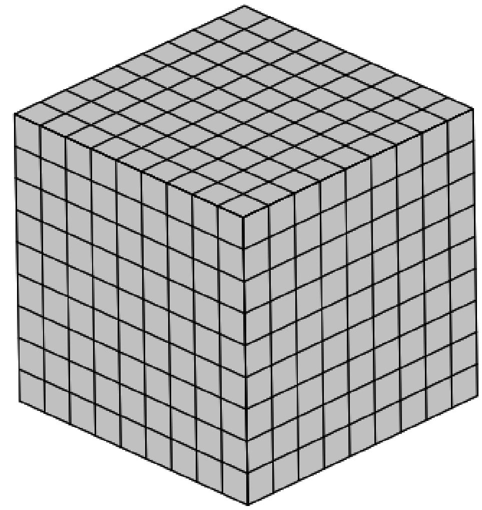
$$\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx \tag{7}$$

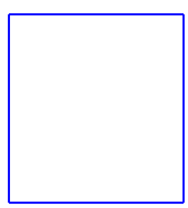
so a triple integral - surface to surface, then curve to curve, then point to point

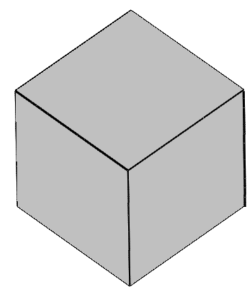
$$\int_{P_1}^{P_2} \int_{C_1}^{C_2} \int_{S_1}^{S_2} f(x,y,z)dV \tag{8}$$

so what about this dV?









$$\begin{aligned} dV &= dzdxdy = dzdydx \\ &= dydxdz = dydzdx \\ &= dxdydz = dxdzdy \end{aligned} \tag{9}$$

For example

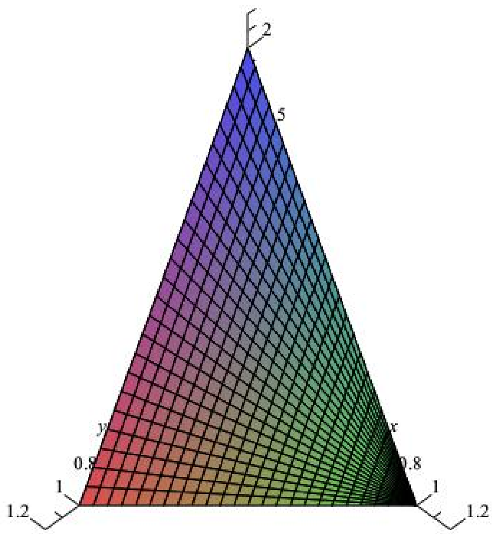
$$\int_{P_1}^{P_2} \int_{C_1}^{C_2} \int_{S_1}^{S_2} f(x, y, z) dzdxdy \tag{10}$$

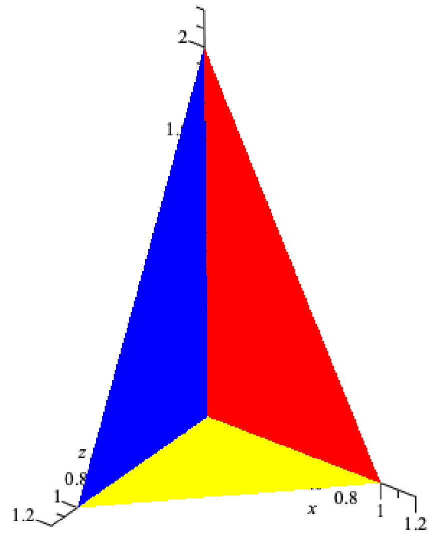
the inside integral, the surface to surface integral, is going in the z direction.

Example 1 Set up the 6 integrals for

$$\iiint_V F(x,y,z)dV \tag{11}$$

where V is the volume bound by $x=0$, $y=0$, $z=0$ and $2x+2y+z=2$



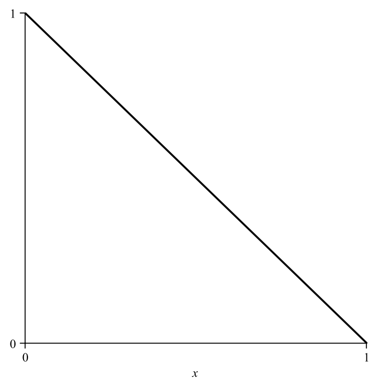


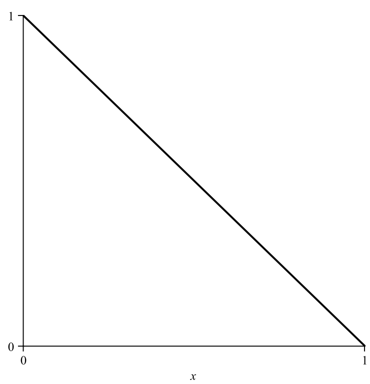
Top View

The surface to surface in going in the z direction so

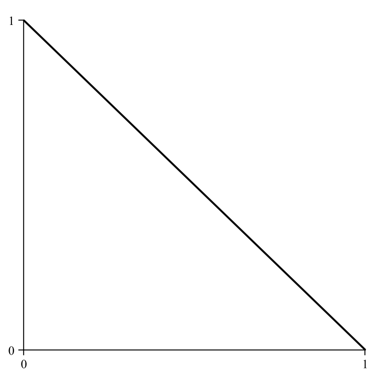
$$\iint_{R_{xy}} \int_0^{2-2x-2y} F(x,y,z)dz\, dA_{xy} \tag{12}$$

When we look straight down we what color do we see





$$\int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} F(x,y,z)dzdydx$$



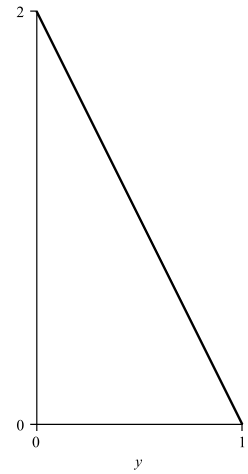
$$\int_0^1 \int_0^{1-y} \int_0^{2-2x-2y} F(x,y,z)dzdxdy$$

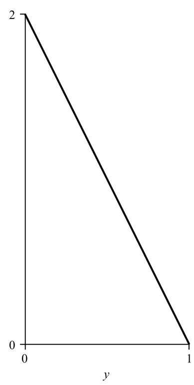
Front View

The surface to surface in going in the x direction so

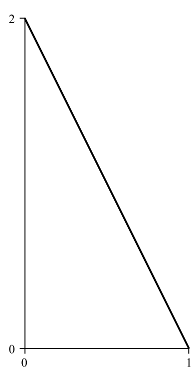
$$\iint\limits_{R_{yz}} \int_0^{\frac{2-2y-z}{2}} F(x,y,z)dx\, dA_{yz} \tag{13}$$

When we look straight down we what color do we see





$$\int_0^1 \int_0^{2-2y} \int_0^{\frac{2-2y-z}{2}} F(x,y,z)\,dx\,dz\,dy$$



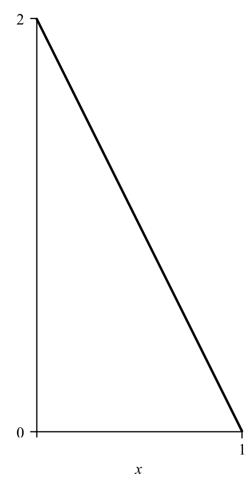
$$\int_0^2 \int_0^{\frac{2-z}{2}} \int_0^{\frac{2-2y-z}{2}} F(x,y,z)\,dx\,dy\,dz$$

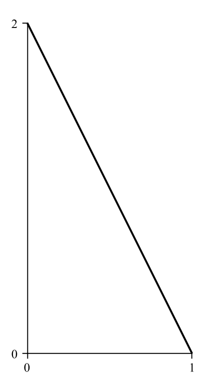
Side View

The surface to surface in going in the y direction so

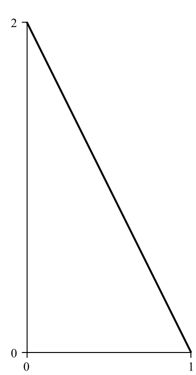
$$\iint_{R_{xz}} \int_0^{\frac{2-2x-z}{2}} F(x,y,z)dy\ dA_{xz} \tag{14}$$

When we look straight down we what color do we see





$$\int_0^1 \int_0^{2-2x} \int_0^{\frac{2-2x-z}{2}} F(x,y,z)dydzdx$$

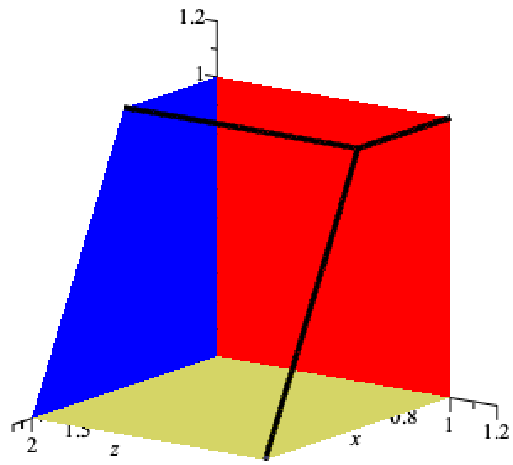


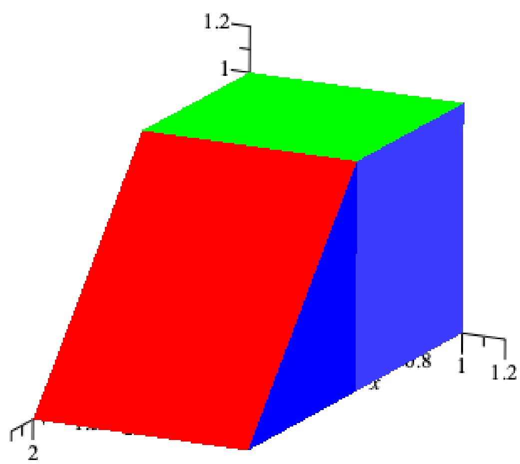
$$\int_0^2 \int_0^{\frac{2-z}{2}} \int_0^{\frac{2-2x-z}{2}} F(x,y,z)dydxdz$$

Example 2 Set up the 6 integrals for

$$\iiint\limits_V F(x,y,z)dV \tag{15}$$

where V is the volume bound by the planes $x = 0$, $y = 0$, $z = 0$, $z = 1$, $y = 1$ and $x + z = 2$



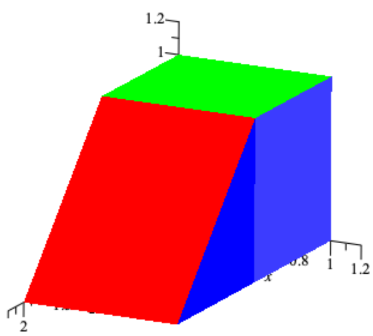


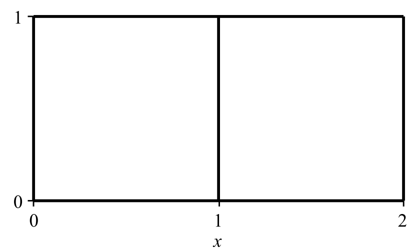
Top View

The surface to surface in going in the z direction so

$$\iint_{R_{xy}} \int_{S_1}^{S_2} F(x,y,z)dz\, dA_{xy} \tag{16}$$

When we look straight down we what color do we see





$$\int_0^1 \int_0^1 \int_0^1 F(x,y,z)dzdydx + \int_1^2 \int_0^1 \int_0^{2-x} F(x,y,z)dzdydx$$

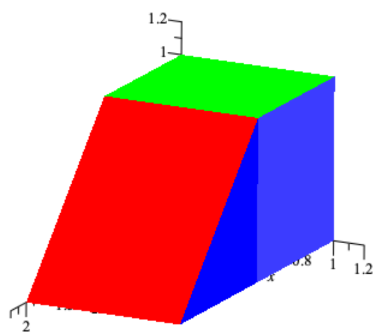
$$\int_0^1 \int_0^1 \int_0^1 F(x,y,z)dzdxdy + \int_0^1 \int_1^2 \int_0^{2-x} F(x,y,z)dzdxdy$$

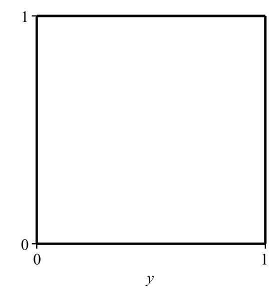
Front View

The surface to surface in going in the x direction so

$$\iint_{R_{yz}} \int_0^{2-z} F(x,y,z)dx\, dA_{yz} \tag{17}$$

When we look straight down the x axis we what color do we see





$$\int_0^1 \int_0^1 \int_0^{2-z} F(x,y,z)dxdzdy$$

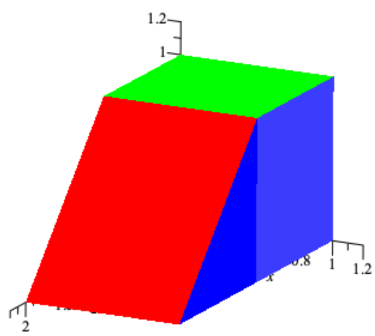
$$\int_0^1 \int_0^1 \int_0^{2-z} F(x,y,z)dxdydz$$

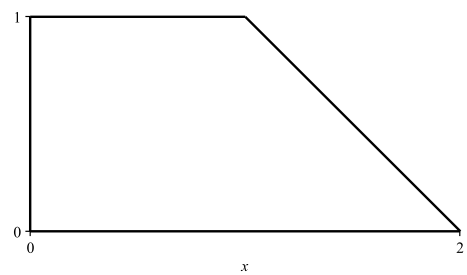
Side View

The surface to surface in going in the y direction so

$$\iint_{R_{xy}} \int_0^1 F(x,y,z)dy\, dA_{xz} \tag{18}$$

When we look straight down the y axes we what color do we see?





$$\int_0^1 \int_0^1 \int_0^1 F(x,y,z)dydzdx + \int_1^2 \int_0^{2-x} \int_0^1 F(x,y,z)dydzdx$$

$$\int_0^1 \int_0^{2-z} \int_0^1 F(x,y,z)dydxdz$$