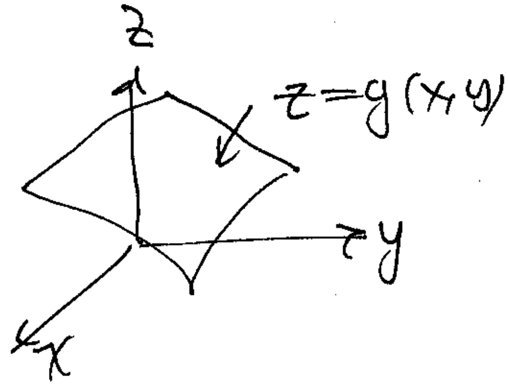
Math 2471 - Calc III

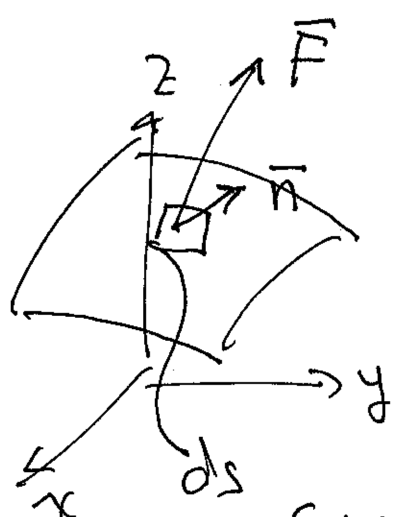
Surface integrals



$$\iint_S f(x,y,z)\,dS$$

$$dS = \sqrt{1+g_x^2+g_y^2}\,dA$$

Now we calculate surface integrals over vector fields - "flux"



$\bar{F}$ - vector field

$\bar{n}$ - normal (unit)

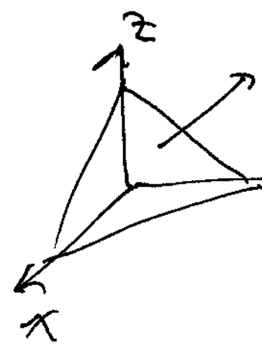
so $\bar{F}\cdot\bar{n}\,ds$

Suppose $\bar{F}$ is the flow of a fluid we want to compute the flow through the surface $\bar{F}\cdot\bar{n}$ is flow at some point

$\bar{F}\cdot\bar{n}\,ds$ flow through some small surface element

&, $\iint_S \vec{F}\cdot\vec{n}\,dS$ flow a flux through entire surface

__Ex 1__ Find the flux of the vector field $\vec{F} = \langle y, x, 1\rangle$ through the surface $x+y+z=1$ where $x, y, z \geq 0$



$\vec{n} = \langle 1, 1, 1\rangle$

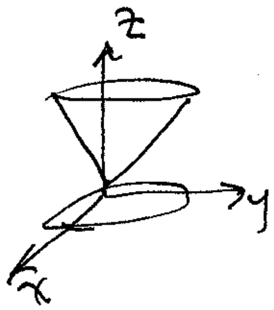
unit normal $\hat{n} = \dfrac{\langle 1, 1, 1\rangle}{\sqrt{3}}$

$z = 1-x-y \quad z_x = -1,\ z_y = -1 \quad dS = \sqrt{1+1+1}\,dA$

so $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \langle y, x, 1\rangle \cdot \frac{\langle 1, 1, 1\rangle}{\sqrt{3}}\sqrt{3}\,dA$

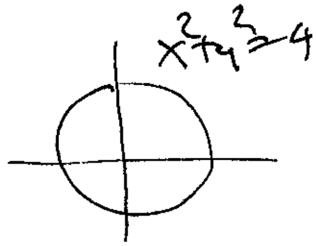
$\displaystyle\int_0^1\int_0^{1-x}(1+x+y)\,dy\,dx = \frac{5}{6}$

Ex 2 Find the flux of the VF $\vec{F} = \langle y, -x, z \rangle$ through the surface $z = \sqrt{x^2+y^2}$ $0 \le z \le 2$



$\vec{n} = \langle -g_x, -g_y, 1 \rangle$

so $g_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $g_y = \dfrac{y}{\sqrt{x^2+y^2}}$



$\vec{n} = \left\langle \dfrac{-x}{\sqrt{x^2+y^2}}, \dfrac{-y}{\sqrt{x^2+y^2}}, 1 \right\rangle$ this is inward

$\vec{n} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$ this is outward

Q is this a unit normal?

$$\|\vec{n}\| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} = \sqrt{1+1} = \sqrt{2}$$

So $\hat{n} = \left\langle \dfrac{x}{\sqrt{2}\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{2}\sqrt{x^2+y^2}}, \dfrac{1}{\sqrt{2}} \right\rangle$

$z = \sqrt{x^2+y^2}$ $\quad dS = \sqrt{1+g_x^2+g_y^2}\, dA = \sqrt{1 + \dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2}}\, dA$

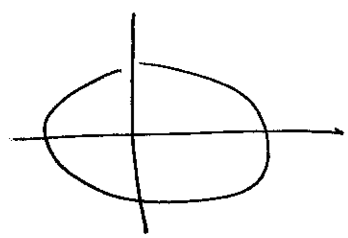
so $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,dS$

$$\iint_R \langle y, -x, z\rangle \cdot \left\langle \frac{x}{\cancel{\sqrt{2}}\sqrt{x^2+y^2}}, \frac{y}{\cancel{\sqrt{2}}\sqrt{x^2+y^2}}, \frac{-1}{\cancel{\sqrt{2}}} \right\rangle \cancel{\sqrt{2}}\,dA$$

$$\iint_R \left(\cancel{\frac{xy}{\sqrt{x^2+y^2}}} - \cancel{\frac{xy}{\sqrt{x^2+y^2}}} - z \right) dA$$

↑ bring in surface

$$- \iint_R \sqrt{x^2+y^2}\,dA$$



go polar

$$-\int_0^{2\pi}\int_0^{2} r\cdot r\,dr\,d\theta = -\int_0^{2\pi} \frac{r^3}{3}\Big|_0^2 d\theta$$

$$= -\frac{8}{3}\int_0^{2\pi} d\theta = -\frac{8}{3}\cdot\theta\Big|_0^{2\pi} = -\frac{16\pi}{3}$$

so flux is into surface.