Calculus 3 - Vector Review

A vector is a directed line segment. For example, consider the points $P(1,1)$ and $Q(2,3)$. The line connecting $P \to Q$ is the vector (see figure 1). Note we have an arrow to denote it has direction.

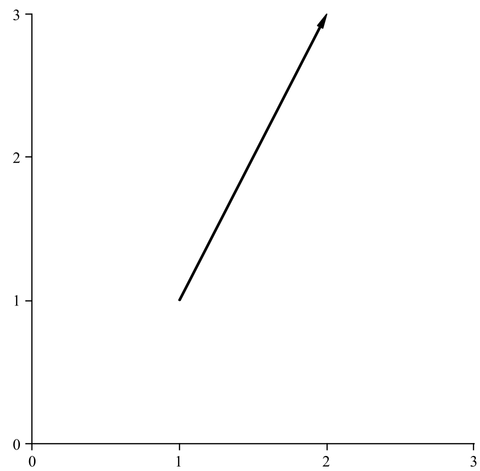


Figure 1: A vector

<u>*Symbolically*</u>

The vector in this case is $\vec{u} = \overrightarrow{PQ} =< 2-1, 3-1 >=< 1,2 >$. Notice the arrow over the vector and also notice we use angle brackets $< 1,2 >$ instead of $(1,2)$ which would be a point.

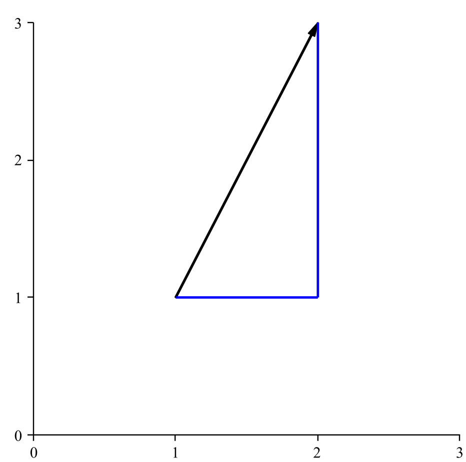


Figure 2: Same vector

Magnitude

Consider the vector $\vec{u} =< 4, 3 >$ (see figure 3). The length of this vector

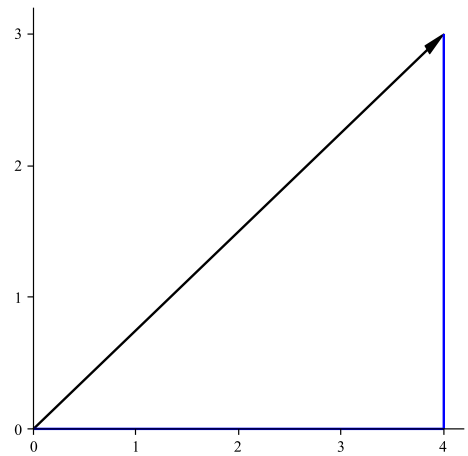


Figure 3: Magnitude

is the length of the hypotenuse of the triangle with sides of length 3 and 4. So using the pythagorean theorem we have

$$\|\vec{u}\| = \sqrt{3^2 + 4^2}$$

noting the we are using the symbol $\|\cdot\|$ for magnitude. With general vectors, say $\vec{u} =< u_1, u_2 >$ or $\vec{u} =< u_1, u_2, u_3 >$ then

$$\begin{aligned} \|\vec{u}\| &= \sqrt{u_1^2 + u_2^2} \ \ (2D) \\ &= \sqrt{u_1^2 + u_2^2 + u_3^2} \ \ (3D) \end{aligned} \tag{1}$$

Scalar Multiplication

If $\vec{u} =< u_1, u_2 >$ and c is some number (not zero) then

$$c\vec{u} = c < u_1, u_2 > = < cu_1, cu_2 > . \tag{2}$$

If $c > 1$ the vector stretches, if $0 < c < 1$ the vector shortens, and if $c < 0$ the vector is sent in the opposite direction.

Zero Vector $-$ A vector that has no length $\vec{0} = < 0, 0 >$.

Vector Addition and Subtraction

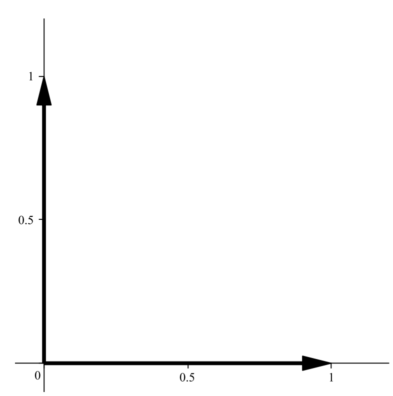
Consider the two vector $\vec{u} = < u_1, u_2 >$ and $\vec{v} = < v_1, v_2 >$ then

$$\begin{aligned} \vec{u} + \vec{v} &= < u_1 + v_1, u_2 + v_2 > \\ \vec{u} - \vec{v} &= < u_1 - v_1, u_2 - v_2 > \end{aligned} \tag{3}$$

Base Vectors

We identify two special vectors in 2D. These are

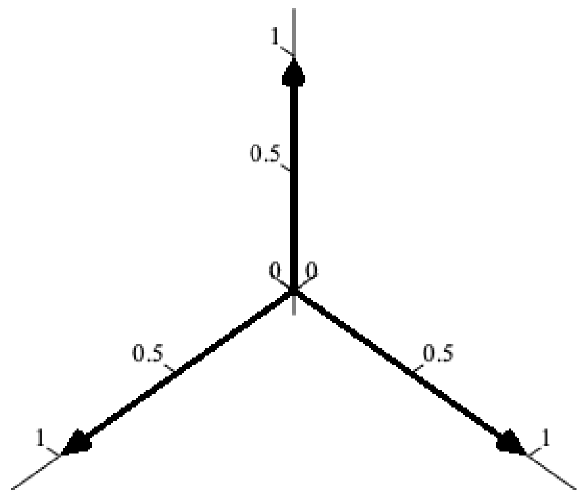
$$\vec{i} = < 1, 0 > \quad \text{and} \quad \vec{j} = < 0, 1 >$$



They are special because they are unit vectors, they are perpendicular to each other and all 2D vectors can be written as a combination of these two.

We also have an analogous result for 3D. So

$$\vec{i} = < 1, 0, 0 > \quad \vec{j} = < 0, 1, 0 > \quad \vec{k} = < 0, 0, 1 >$$



Dot Product

The dot product of two vectors $\vec{u} = < u_1, u_2 >$ and $\vec{v} = < v_1, v_2 >$ is

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 \\ &= \|\vec{u}\| \|\vec{v}\| \cos\theta \end{aligned}$$

where θ is the angle between the vectors. In 3D where $\vec{u} = < u_1, u_2, u_3 >$

and $\vec{v} =< v_1, v_2, v_3 >$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

One important feature of this alternate definition is that if

$$\vec{u} \perp \vec{v} \quad \Rightarrow \quad \vec{u} \cdot \vec{v} = 0$$

Cross Product

Given vectors $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$ we define the cross product between then two vector as

$$\vec{u} \times \vec{v} =< u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 >$$

However, this definition is a little hard to use so we'll come up with a better way to calculate cross products.

First we define a determinant (you will see this in Linear Algebra).

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

so for example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = 4 - 6 = -2$$

Now we define the cross product

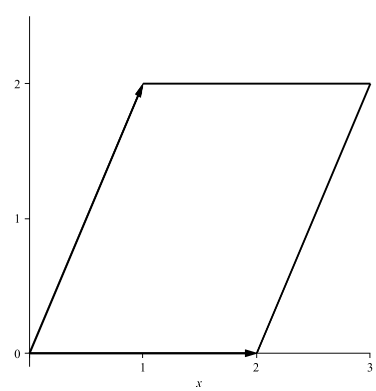
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

We also have the following which will be important in Calc 3.

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$$

which is the area of the parallelogram with $\vec{u}$ and $\vec{v}$ as the sides.



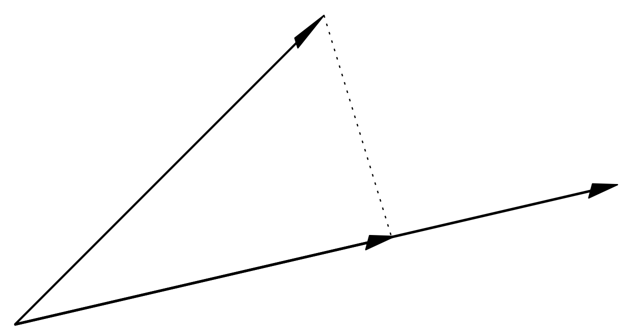
From the figure we see the height h and base of the parallelogram is

$$h = \|\vec{v}\| \sin\theta, \qquad \|\vec{u}\|$$

and multiplying these two together gives the formula.

Projections

Given vectors $\vec{u}$ and $\vec{v}$, we created a formula to project $\vec{v}$ onto $\vec{u}$. This was given by



$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} \tag{4}$$

Lines

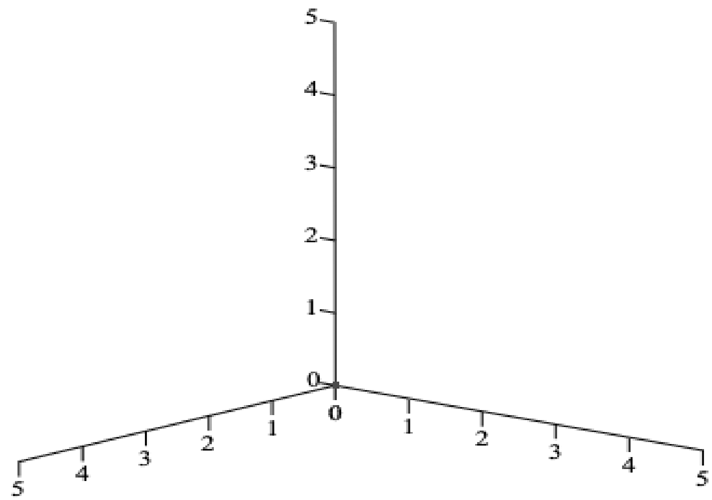
Given a point $P(x_0, y_0, z_0)$ and direction $\vec{u} =< a, b, c >$ the line through P in the direction of $\vec{u}$ is given by

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned} \tag{5}$$

Planes

A plane is simply a flat surface in 3D (like a piece of paper). We characterize this surface by a single vector that is perpendicular to every vector lying on the place. This special vector is called the “normal” denoted by

$$\vec{n} =< a, b, c > \tag{6}$$



If $P(x_0, y_0, z_0)$ is a given point on the plane and $Q(x, y, z)$ which is a point that moves around on the plane, then

$$\overrightarrow{PQ} = < x - x_0, y - y_0, z - z_0 > \tag{7}$$

is a vector lying on the plane and if we dot this with the normal, we get

$$\vec{n} \cdot \overrightarrow{PQ} = < a, b, c > \cdot < x - x_0, y - y_0, z - z_0 > = 0 \tag{8}$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{9}$$

This is called the "normal-point" form. If we expand and moved all the numbers to the right side $ax_0 + by_0 + cz_0 = d$ then we have the form

$$ax + by + cz = d \tag{10}$$

called the general form.