Math 3331 ODE's

(1) Separable

$$\frac{dy}{dx} = f(x)g(y)$$

separate $\displaystyle\int \frac{dy}{g(y)} = \int f(x)dx + c$

(2) linear

$$\frac{dy}{dx} + p(x)y = q(x)$$

integrating factor $\mu = e^{\int p(x)dx}$

ODE becomes

$$\frac{d}{dx}(\mu y) = \mu q \quad \text{sep. \& } \int$$

(3) Bernoulli

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

divide by y^n $\quad \displaystyle\frac{1}{y^n}\frac{dy}{dx} + p(x)\left(\frac{y}{y^n}\right) = q(x)$ $\quad$ let $u = \dfrac{y}{y^{n-1}}$

(14) Riccati

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x)$$

if we know 1 sol^n then y_1

$$y = y_1 + \frac{1}{u}$$

turn the ODE into a linear ODE.

Ex $$\frac{dy}{dx} = y^2 - 2xy + x^2 + 1$$

1 sol^n $y = x$

let $y = x + \frac{1}{u}$, $y' = 1 - \frac{u'}{u^2}$

sub
$$1 - \frac{u'}{u^2} = \left(x + \frac{1}{u}\right)^2 - 2x\left(x + \frac{1}{u}\right) + x^2 + 1$$

$$\cancel{1} - \frac{u'}{u^2} = \cancel{x^2} + \frac{2x}{u} + 1/u^2 - \cancel{2x^2} - \frac{2x}{u} + \cancel{x^2} + \cancel{1}$$

$$-\frac{u'}{u^2} = \cancel{\frac{2x}{u}} + \frac{1}{u^2} - \cancel{\frac{2x}{u}}$$ more cancellation

New ODE

$$u' = -1 \quad \text{Real easy}$$

so $\quad u = -x + c$

sol^n $\quad y = x + \frac{1}{u} = x + \frac{1}{c-x}$

<u>Ex2</u> $\quad y' = e^{-x} y^2 + y - 4e^{x}$

1 sol^n $\quad y = 2e^{x}$

y check $\quad y = Ke^{x}, \; y' = Ke^{x}$

$$Ke^{x} = e^{-x} K^2 e^{2x} + Ke^{x} - 4e^{x}$$

$$K\cancel{e^{x}} = K^2 \cancel{e^{x}} + K\cancel{e^{x}} - 4\cancel{e^{x}}$$

$$K^2 - 4 = 0 \Rightarrow K = \pm 2$$

$y = 2e^x + \frac{1}{u}$, $\quad y' = 2e^x - \frac{u'}{u^2}$

sub

$$2e^x - \frac{u'}{u^2} = e^{-x}\left(2e^x + \frac{1}{u}\right)^2 + \left(2e^x + \frac{1}{u}\right) - 4e^x$$

$$= e^{-x}\left(4e^{2x} + \frac{4e^x}{u} + \frac{1}{u^2}\right) + 2e^x + \frac{1}{u} - 4e^x$$

$$\cancel{2e^x} - \frac{u'}{u^2} = \cancel{4e^x} + \frac{4}{u} + \frac{e^{-x}}{u^2} + \cancel{2e^x} + \frac{1}{u} - \cancel{4e^x}$$

Expect e^x's to cancel

so $\quad -\frac{u'}{u^2} = \frac{5}{u} + \frac{e^{-x}}{u^2}$

$$u' = -5u - e^{-x} \quad \text{linear}$$

$$u' + 5u = -e^{-x} \quad \text{SF linear}$$

$$\mu = e^{5x} \Rightarrow \frac{d}{dx}\left(e^{5x}u\right) = -e^{5x} \cdot e^{-x} = -e^{4x}$$

$$e^{5x} u = -\frac{1}{4} e^{4x} + c$$

$$u = -\frac{1}{4} e^{-x} + c e^{-5x}$$

sol^n

$$y = 2e^{x} + \frac{1}{-\frac{1}{4} e^{-x} + c e^{-5x}}$$

if we had an I.C. $y(0) = 0$ we would use this to find c. So

$$0 = 2 + \frac{1}{-\frac{1}{4} + c}$$

$$\Rightarrow \quad \frac{1}{c - \frac{1}{4}} = -2 \quad \Rightarrow \quad c - \frac{1}{4} = -\frac{1}{2} \quad \Rightarrow \quad c = -\frac{1}{4}$$

so

$$y = 2e^{x} + \frac{1}{-\frac{1}{4} e^{-x} - \frac{1}{4} e^{-5x}}$$

$$= 2e^{x} - \frac{4}{e^{-x} + e^{-5x}}$$

Final note: Guessing 1 soln

This can be difficult. One way

compare terms.

Ex $y' = y^2 - \frac{y}{x} - \frac{1}{x^2}$ (HW)

Compare $y^2 = \frac{y}{x} \Rightarrow y = \frac{1}{x}$

$-\frac{y}{x} = -\frac{1}{x^2} \Rightarrow y = \frac{1}{x}$

$y^2 = +\frac{1}{x^2} \Rightarrow y = \pm\frac{1}{x}$ (forget sign)

so try $y = k/x$ & see if there is a k