Product Series and Sum Series

for kπ where 0≤k≤π

Product Series for Cos(kπ), Sin(kπ) and Tan(kπ)

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Abstract

The following identity is developed by placing a square inside a circle and repeatedly bisecting the chords that exists in quadrant 1 between (0,1) and (1,0)

$$\mathbf{lim}_{2^n\to\infty} \mathbf{2^n}\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}} = \frac{\pi}{2}; \textit{where } \langle n\rangle \textit{ is number of nested terms}$$

And further yielding the trig identities $\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}} = \mathbf{2}\,\mathbf{sin}\left({}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n+2}}\right)$

$$\sqrt{\mathbf{2}+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}} = \mathbf{2}\,\mathbf{cos}\left({}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n+2}}\right)$$

$$\frac{\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}}{\sqrt{\mathbf{2}+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}} = \mathbf{tan}\left({}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n+2}}\right)$$

$$\lim_{2^n\to\infty} \mathbf{2^n}\,\mathbf{tan}\,{}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n}} = \boldsymbol{\pi}$$

$$\lim_{2^n\to\infty} \mathbf{2^n}\,\mathbf{sin}\,{}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n}} = \boldsymbol{\pi}$$

Further analysis of any chord $\boldsymbol{S_0}$: $\mathbf{lim}_{2^n\to\infty} \mathbf{2^n}\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{4}-\boldsymbol{S_0}^2}}}} = \boldsymbol{k\pi}\ ; 0 \le k \le 1$

$S_0{}^2 = (\Delta x)^2 + (\Delta y)^2$ for any chord on the unit circle such that S_0 is the line segment between (x_1, y_1) *and* (x_2, y_2). Where $x \to \Re$ *and* $y \to \Im$. For $\theta = 0$; $(1,0)$ *and* $(x_2, \hat{\imath}y_2)$ and $S_0{}^2 = (1 - x_2)^2 + (y_2)^2$

Further trig analysis yields:

$$\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}} = \mathbf{2}\,\mathrm{cos}\left({}^{\mathbf{k\pi}}/_{\mathbf{2}^{n}}\right)$$

$$\sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - (S_0)^2}}}} = \mathbf{2} \sin\left(\mathbf{k\pi}/_{\mathbf{2^{n+1}}}\right)$$

$$\frac{\sqrt{2 + \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{4 - (S_0)^2}}}}}{\sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - (S_0)^2}}}}} = \tan\left(\mathbf{k\pi}/_{\mathbf{2^{n+1}}}\right)$$

By circumscribing the circle inside the square and collapsing the square onto the circle we approach the limit of π from the other side:

$$\lim_{2^n \to \infty} 2^n \frac{\sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}}{\sqrt{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}} = \frac{\pi}{2}$$

$$\lim_{2^n \to \infty} 2^n \frac{\sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{\mathbf{4} - \boldsymbol{S_0}^{\mathbf{2}}}}}}}{\sqrt{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{\mathbf{4} - \boldsymbol{S_0}^{\mathbf{2}}}}}}} = k\pi$$

$$\lim_{2^n \to \infty} 2^{n+2} \frac{\sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}}{\sqrt{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}} = \pi$$

By analyzing the triangular area in each quadrant when a square is circumscribe within the unit circle and adding the recursive triangular fractal areas we arrive at:

$$\pi = \sum_{1}^{\frac{ln\infty}{ln2}-1} 2 + 2^{n-1}\left(2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \sqrt{2 + \cdots\sqrt{2}}}}\right)\left(\sqrt[\langle n \rangle]{2 - \sqrt{2 + \sqrt{2 + \cdots\sqrt{2}}}}\right)$$

$$k\pi = \sum_{1}^{\frac{ln\infty}{ln2}} \frac{y}{2} + 2^{n-3}\left(2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \sqrt{2 + \cdots\sqrt{4 - {S_\theta}^2}}}}\right)\left(\sqrt[\langle n \rangle]{2 - \sqrt{2 + \sqrt{2 + \cdots\sqrt{4 - {S_\theta}^2}}}}\right);\quad 0 \le k \le {}^1\!/_2$$

S_θ is the chord length where θ corresponds to $2k\pi$ (Since area $\frac{\pi}{4}$ requires $\theta = \frac{\pi}{2}$ therefore $S_\theta = \sqrt{2}$)

By inscribing the circle within a square and subtracting the recursive triangular fractal areas we arrive at

$$\frac{\pi}{2} = 2 - \sum_{1}^{2^n=\infty} 2^n \left(\frac{2 - \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots\sqrt{\mathbf{2}}}}}{\sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots\sqrt{\mathbf{2}}}}}\right)\left(\sqrt{\frac{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots\sqrt{2}}}}{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots\sqrt{2}}}}}\right)$$

$$k\pi = \frac{2y}{x} - \sum_{1}^{2^n=\infty} 2^n \left(\frac{2 - \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots\sqrt{4 - (S_0)^2}}}}{\sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots\sqrt{4 - (S_0)^2}}}}\right)\left(\sqrt{\frac{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots\sqrt{4 - (S_0)^2}}}}{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots\sqrt{4 - (S_0)^2}}}}}\right)$$

$$x, y \; for \; S_0 \; from \; \frac{k\pi}{2}$$

The work introduces the term Quasi-Infinite series as the series reaches absolute convergence before ∞. In fact the series will be demonstrated to reach absolute convergence at $n \cong \frac{\ln\infty}{\ln 2}$.

The development relies on the concept of orthogonality. The initial inner product space is created relying on $\Im \perp \Re$ and $\Im \cdot \Re = 0$. However it becomes obvious that an "interval orthogonality' exists. This interval is defined by the number of orthogonal steps. Each step relies on Pythagoras to calculate the chord length or recursive fractal area necessary to equate the radial distance traveled $d\theta$, in terms of Euclidean orthogonal steps $\frac{dy}{dx}$. As the number of steps or intervals increases the interval spacing approaches zero while $\frac{dy}{dx} \to \infty$.

At this point we have what the author refers to as orthogonal growth. The concept of real and orthogonal growth will be further examined to determine if a dynamical physical space can result from the resultant spin of orthogonal growth. Hence one dimensional real growth may be able to create multidimensional space. The mathematical concept would reconcile quantum physics with string theory.

We will further introduce the postulate that absolute convergence occurs only if the series converges at $n < \infty$. In fact we will present the possibility that $n \cong f(\ln \infty)$ may be the limit for absolute convergence.

The use of trig identities to prove absolute convergences generates the Euler identity. The analysis of this identity and the recurrence of the nested 2's also indicate that the Sine and Cosine Transforms may be useful in analysing this space. This may have been the impetus to the 2003 paper by Servi commissioned by The USA Air Force where Servi used the Fourier Transform to develop his table of trig identities using the nested 2.

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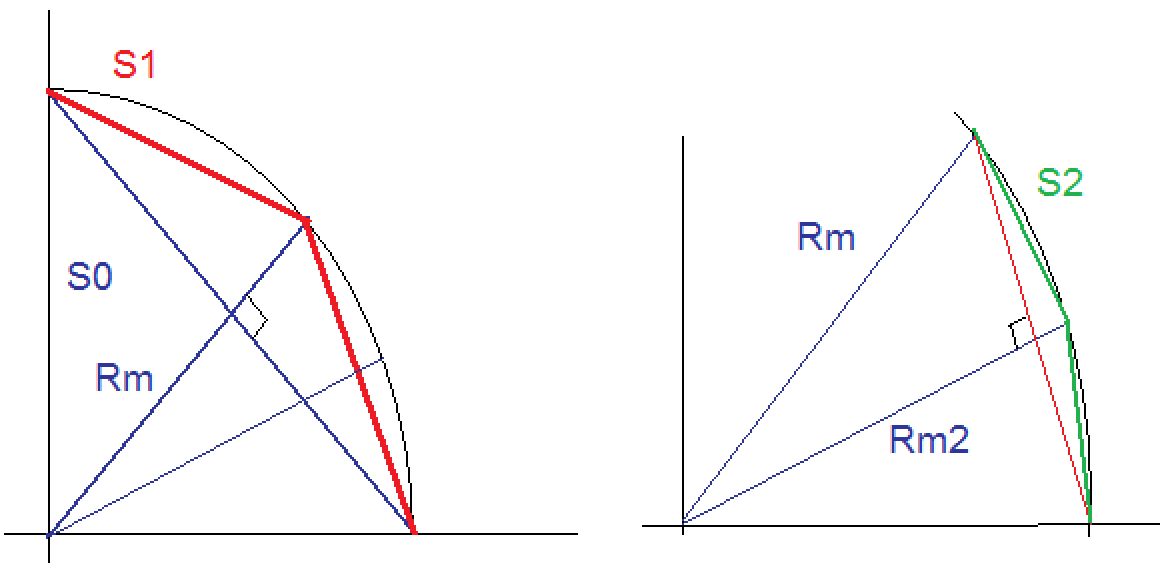
Inscribed Square derivation of π

Derivation for Inscribed Square Product Series approximation of $\pi/2$

$$(ID\ 1) \qquad \lim_{2^n \to \infty} 2^n \sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}} = \frac{\pi}{2};$$

where $\langle n \rangle$ is number of nested terms



Calculating the length of the sides as we continue to half the arc length. S_n corresponds to the chord length when 2^n corresponds to the number of chords in $\frac{\pi}{2}$ arc length.

$$S_0 = \sqrt{2}$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + (1 - Rm)^2}$$

$$Rm = \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}\right)^2}$$

$$S_1 = \sqrt{2 - 2\sqrt{1 - \left(\frac{S_0}{2}\right)^2}}$$

$$S_1 = \sqrt{2 - \sqrt{4 - {S_0}^2}}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + (1 - Rm2)^2}$$

$$Rm2 = \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}\right)^2}$$

$$S_2 = \sqrt{2 - \sqrt{4 - {S_1}^2}}$$

$$S_3 = \sqrt{\left(\frac{S_2}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_2}{2}\right)^2}\right)^2}$$

$$S_3 = \sqrt{2 - \sqrt{4 - {S_2}^2}}$$

$$S_3 = \sqrt{2 - \sqrt{4 - 2 + \sqrt{4 - 2 + \sqrt{4 - 2}}}}$$

Given that 2^n is the number of "orthogonal steps" or chords circumscribed by $\frac{\pi}{2}$ radians and S_n

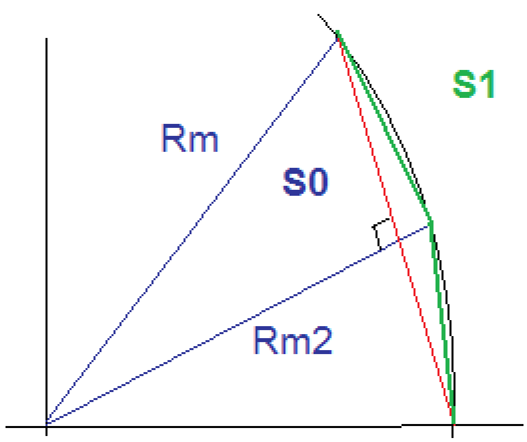
is the length of side n the following becomes obvious:

$$\lim_{2^n \to \infty} 2^n \sqrt{2 - \overset{\langle n \rangle}{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}} = \frac{\pi}{2}$$

Although proof of convergence and the point or value of convergence should not be necessary since the model is to approximate arc length $\frac{\pi}{2}$, simple algebraic manipulation of Viete or Euler should provide sufficient rigor to prove absolute convergence. I am not aware of any other series approximations for Pi that do not rely on some type of mathematical equation and for this reason I believe this series derivation may be significant.

Derivation of the Product Series approximation for any

Radial arc length between 0 and π



For any $k\pi$ chord length is S_0

Calculating the length of the sides as we continue to half the arc length. S_n corresponds to the chord length when 2^n corresponds to the number of chords in $k\pi$ arc length.

$$S_0 = chord\ length$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + (1 - Rm)^2}$$

$$Rm = \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}\right)^2}$$

$$S_1 = \sqrt{2 - 2\sqrt{1 - \left(\frac{S_0}{2}\right)^2}}$$

$$S_1 = \sqrt{2 - \sqrt{4 - {S_0}^2}}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + (1 - Rm2)^2}$$

$$Rm2 = \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}\right)^2}$$

$$S_2 = \sqrt{2 - \sqrt{4 - {S_1}^2}}$$

$$S_3 = \sqrt{\left(\frac{S_2}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_2}{2}\right)^2}\right)^2}$$

$$S_3 = \sqrt{2 - \sqrt{4 - {S_2}^2}}$$

$$S_3 = \sqrt{2 - \sqrt{4 - 2 + \sqrt{4 - {S_0}^2}}}$$

Given that 2^n is the number of steps or chords circumscribed by $k\pi$ radians and S_n is the length of side n : $\lim_{2^n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{4 - {S_0}^2}}}}} = k\pi$

Series approximation for any radial arc length between 0 and π

$$(ID\ 2) \qquad \lim_{2^n \to \infty} 2^n \sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - {S_0}^2}}}} = k\pi$$

where $\langle n \rangle$ is number of nested terms and So is the chord length of $k\pi$

where $0 \le k \le 1$

In general ${S_0}^2 = (\Delta x)^2 + (\Delta y)^2$ for any chord on the unit circle such that S_0 is the line segment between (x_1, y_1) *and* (x_2, y_2). Where $x \to \Re$ *and* $y \to \Im$.

For $\theta = 0$; $(1,0)$ *and* $(x_2, \hat{\imath} y_2)$ and ${S_0}^2 = (1 - x_2)^2 + (y_2)^2$

$$\boldsymbol{iff}\ \Im \perp \Re$$

It is the perpendicular nature of $\Im$ relative to the $\Re$ axis that forces Pythagoras for ${S_0}^2 = (\Delta x)^2 + (\Delta y)^2$ and it is this last term that ultimately predicts arc length in Euclidean space.

Therefore:

$$(ID\ 2a) \qquad \lim_{2^n\to\infty} 2^n\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-(1-x_2)^2+\left(y_2\right)^2}}}}=k\pi$$

where $\langle n\rangle$ *is number of nested terms and* $(1-x_2)^2+(y_2)^2$ *is the chord length* of $k\pi$

where $0\le k\le 1$

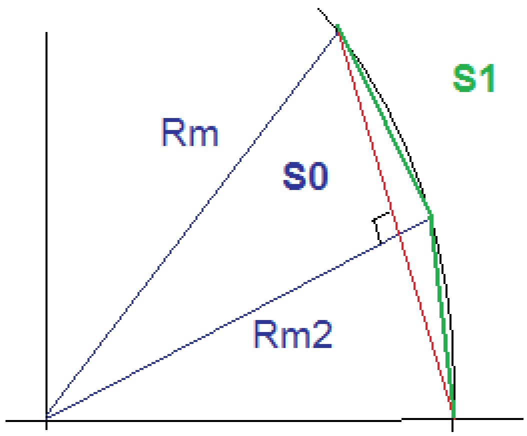
$$\boldsymbol{iff}\ \mathfrak{I}\ \perp\ \mathfrak{R}$$

It is the perpendicular nature of $\mathfrak{I}$ relative to the $\mathfrak{R}$ axis that forces Pythagoras for ${S_0}^2=(\Delta x)^2+(\Delta y)^2$ and it is this last term that ultimately predicts arc length in Euclidean space.

Example of Series approximation of $\frac{\pi}{3}$

$$(ID\ 2b) \qquad \lim_{2^n\to\infty} 2^n\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{3}}}}=\frac{\pi}{3};$$

where $\langle n\rangle$ *is number of nested terms*



For $\frac{\pi}{3}$ or 60 degrees the chord length, $S_0 = 1$

Calculating the length of the sides as we continue to half the arc length. S_n corresponds to the chord length when 2^n corresponds to the number of chords in $\frac{\pi}{2}$ arc length.

$$S_0 = 1$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + (1 - Rm)^2}$$

$$Rm = \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}$$

$$S_1 = \sqrt{\left(\frac{S_0}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2}\right)^2}$$

$$S_1 = \sqrt{2 - 2\sqrt{1 - \left(\frac{S_0}{2}\right)^2}}$$

$$S_1 = \sqrt{2 - \sqrt{4 - 1^2}}$$

$$S_1 = \sqrt{2 - \sqrt{3}}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + (1 - Rm2)^2}$$

$$Rm2 = \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}$$

$$S_2 = \sqrt{\left(\frac{S_1}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_1}{2}\right)^2}\right)^2}$$

$$S_2 = \sqrt{2 - \sqrt{4 - {S_1}^2}}$$

$$S_3 = \sqrt{\left(\frac{S_2}{2}\right)^2 + \left(1 - \sqrt{1^2 - \left(\frac{S_2}{2}\right)^2}\right)^2}$$

$$S_3 = \sqrt{2 - \sqrt{4 - {S_2}^2}}$$

$$S_3 = \sqrt{2 - \sqrt{4 - 2 + \sqrt{4 - 2 + \sqrt{4 - 1}}}}$$

Given that 2^n is the number of steps or chords circumscribed by $\frac{\pi}{3}$ radians and S_n is the length of side n the following becomes obvious:

$$\lim_{2^n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{3}}}}} = \frac{\pi}{3}$$

Example of Series approximation $2\pi/3$

$$\lim_{2^n\to\infty} 2^n \sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-{S_0}^2}}}} = k\pi$$

where $\langle n\rangle$ *is number of nested terms and So is the chord length* of $k\pi$

where $0 \le k \le 1$

For example chord length of $\frac{2\pi}{3}$ is $\sqrt{3}$ therefore $S_0 = \sqrt{3}$

$$\lim_{2^n\to\infty} 2^n \sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-\left(\sqrt{3}\right)^2}}}} = \frac{2\pi}{3}$$

$$(ID\ 2c) \qquad \lim_{2^n\to\infty} 2^n \sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{1}}}} = \frac{2\pi}{3}$$

Derivation of Trig identity

$$\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}} = 2\sin\left(\frac{\pi}{2}\Big/2^{n+1}\right)$$

or

$$(ID\ 3) \qquad \sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}} = 2\sin\left(\pi/2^{n+2}\right)$$

If one circumscribes the perimeter in a single quadrant such that each subtended length is equal and then reiteratively circumscribes each subtended arc symmetrically we divide the circumference into equidistance arcs such that

$$d\theta = {}^{\frac{\pi}{2}}\!/_{2^n} = \frac{\pi}{2^{n+1}} rads$$

$$\operatorname{crd} \theta = \sqrt{(1-\cos\theta)^2 + \sin^2\theta} = \sqrt{2-2\cos\theta} = 2\sin\left(\frac{\theta}{2}\right).$$

And the side or arc length or $ds = \sqrt{2 - \sqrt[\langle n \rangle]{2+\sqrt{2+\cdots\sqrt{2}}}} = 2\sin\left({}^{\frac{\pi}{2}}\!/_{2^{n+1}}\right)$

Where $\sqrt[\langle n \rangle]{2+\sqrt{2+\cdots\sqrt{2}}}$ is the nested orthogonality factor and *n* is the number of nested terms.

$$ds = \sqrt{2 - \sqrt[\langle n \rangle]{2+\sqrt{2+\cdots\sqrt{2}}}} = 2\sin {}^{\pi}\!/_{2^{n+2}}$$

In general for any θ where θ is the initial angle and ${}^{\theta}\!/_{2^{n+1}}$ is the bisected angle:

$$(ID\ 3a) \qquad \sqrt{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - \left[(1-x_2)^2 + (y_2)^2\right]}}}} = 2\sin {}^{\theta}\!/_{2^{n+1}}$$

$$where\ [(1-x_2)^2 + (y_2)^2] = S_\theta = 2 - 2\cos\theta$$

Proof for Inscribed Square derivation using Trig Identities

$$\sin 2\theta \; = 2\;\cos\theta \sin\theta$$

$$= 2\;\cos\theta \left[2\;\cos\frac{\theta}{2}\;\;\cdot\sin\frac{\theta}{2}\right]$$

$$\sin\theta \,=\, 2^n \sin\frac{\theta}{2^n}\;\left[\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^n}\right]$$

$$\frac{\sin\theta}{\theta} = \frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\;\left[\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^n}\right]$$

$$n \to \infty, \frac{\theta}{2^n} \to 0, \frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}} \to 1$$

We arrive at Euler's: $\frac{\sin\theta}{\theta} = 1\;\left[\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdots\cos\frac{\theta}{2^n}\right]$

Using trig identity $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}$ and sub $\theta = \frac{\pi}{2}$ into Euler above:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

Inverting Viete we get.

$$\frac{\pi}{2} = 2^n\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2+\sqrt{2}}}\cdot\frac{1}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots$$

Multiplying by the nested conjugate:

$$\frac{\pi}{2} = 2^n\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2+\sqrt{2}}}\cdot\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\cdot\frac{1}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdot\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}\cdot\cdot$$

$$= \frac{2^n}{\sqrt{2}} \cdot \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}} \cdot \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2}}} \cdot \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} \cdots \frac{\sqrt[n-1]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt[\langle n-2\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}$$

$$\cdot \frac{\sqrt[n]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt[\langle n-1\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{2}}}}}$$

$$\frac{\pi}{2} = 2^{n-1}\sqrt{2-\sqrt[\langle n-1\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}$$

$$\frac{\pi}{2} = 2^{n}\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}$$

$$\frac{\sin\theta}{\theta} = 1\left[\cos\frac{\theta}{2} \cdot \sqrt{\frac{1+\cos\frac{\theta}{2}}{2}} \cdots \cos\frac{\theta}{2^n}\right]$$

For any angle θ:

$$\frac{\theta}{\sin\theta} = \frac{2}{2\cos\frac{\theta}{2}} \cdot \frac{2}{\sqrt{2+2\cos\frac{\theta}{2}}} \cdots$$

$$\frac{\theta}{\sin\theta} = 2^n \frac{1}{2\cos\frac{\theta}{2}} \cdot \frac{1}{\sqrt{2+2\cos\frac{\theta}{2}}} \cdot \frac{\sqrt{2-2\cos\frac{\theta}{2}}}{\sqrt{2-2\cos\frac{\theta}{2}}} \cdots$$

$$\frac{\theta}{2\sin\theta} = 2^{n-1} \frac{1}{2\cos\frac{\theta}{2}} \cdot \cdot \frac{\sqrt{2-2\cos\frac{\theta}{2}}}{\sqrt{4-4\cos^2\frac{\theta}{2}}} \cdots$$

For the equation to be valid for any angle θ the following relationship must be true:

$$2\sin\theta = 2\cos\frac{\theta}{2}\sqrt{4 - 4\cos^2\frac{\theta}{2}}$$

$$2\sin\theta = 2\cos\frac{\theta}{2}\,2\sin\frac{\theta}{2}$$

$$\sin\theta = 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}$$

Further the trailing nested term must satisfy both equations, hence

$$\sqrt{2 + 2\cos\frac{\theta}{2}} = \sqrt{4 - \left[(1 - x_2)^2 + \left(y_2\right)^2\right]}$$

$$RS = \sqrt{4 - \left[\left(1 - \cos\frac{\theta}{2}\right)^2 + \left(\sin\frac{\theta}{2}\right)^2\right]}$$

$$= \sqrt{4 - \left[2 - 2\cos\frac{\theta}{2}\right]}$$

$$= \sqrt{2 + 2\cos\frac{\theta}{2}}$$

Therefore the general product series for any θ =kπ

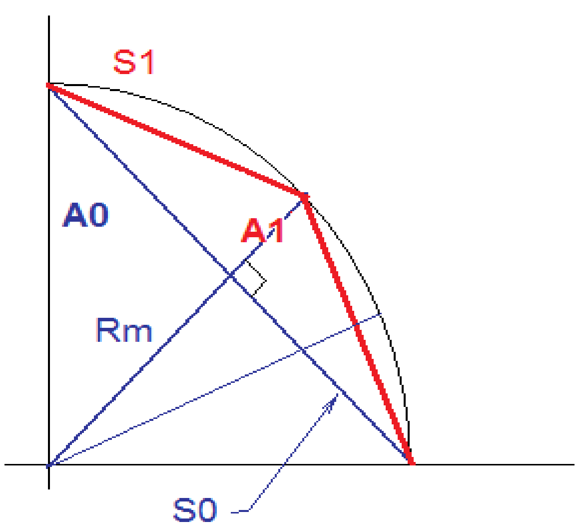
$$\lim_{2^n\to\infty} 2^n \sqrt{2 - \overset{\langle n\rangle}{\sqrt{2 + \sqrt{2 + \cdots \sqrt{4 - \left[(1 - x_2)^2 + \left(y_2\right)^2\right]}}}}} = k\pi$$ satisfies the identity.

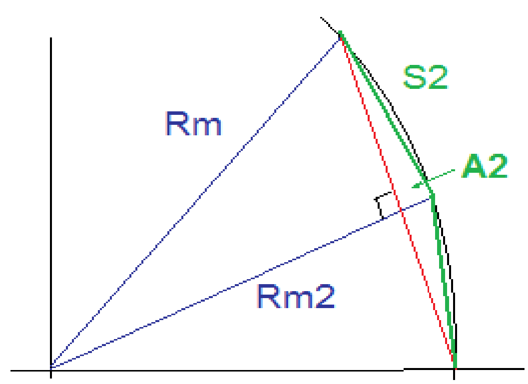
Quasi-Infinite Sum Series for kπ from 0 to π

As seen in prior work, the following series for π converges absolutely at $n \cong \frac{ln\infty}{ln2}$. Hence the series is deemed to be quasi-infinite, given that although the number of terms is a function of infinity it converges far before infinity. It should be noted that the series cannot be said to have an upper limit of $n = \infty$ since the upper limit is defined by the number of "fractal particles" or identical areas added to the previous area, and hence the upper limit is a function of $2^n \to \infty$. More specifically and depending on how the value of n is determined the upper limit could be $2^{n-d} = \infty$, where d is determined by the displacement of the nesting term $\langle n \rangle$ in the series structure and the specific quadrant we are addressing with the series. In general the upper limit can be assumed to be $n = \frac{ln\infty}{ln2}$ given that the displacement results in a small limit change $n = \frac{ln\infty}{ln2} + d$. Any Series deemed to be Quasi-infinite converges and if the point of convergence is less than or equal to $\frac{ln\infty}{ln2}\, th\ term$, it converges absolutely and the sum of the series is known as a Real number.

The following is a specific series sum for the specific value as defined by the area of the unit circle. The initial area of 2 is the largest area of a square that can be inscribed on the unit circle while each consecutive term is the recursive fractal area added by doubling the number of sides of the inscribed equilateral polygon.

$$\pi = \sum_{1}^{\frac{ln\infty}{ln2}-1} 2 + 2^{n-1}\left(2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}\right)\left(\sqrt[\langle n \rangle]{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{2}}}}\right)$$





$$\pi = 2 + (2-\sqrt{2})\sqrt{2} + 2\left(2-\sqrt{2+\sqrt{2}}\right)\left(\sqrt{2-\sqrt{2}}\right) + 4\left(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2}}}\right)$$
$$+ 8\left(2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)$$
$$+ 16\left(2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)\left(\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right) + \cdots$$

kπ from 0 to $\frac{\pi}{2}$

In general S_θ is the chord length where θ corresponds to $2k\pi$.

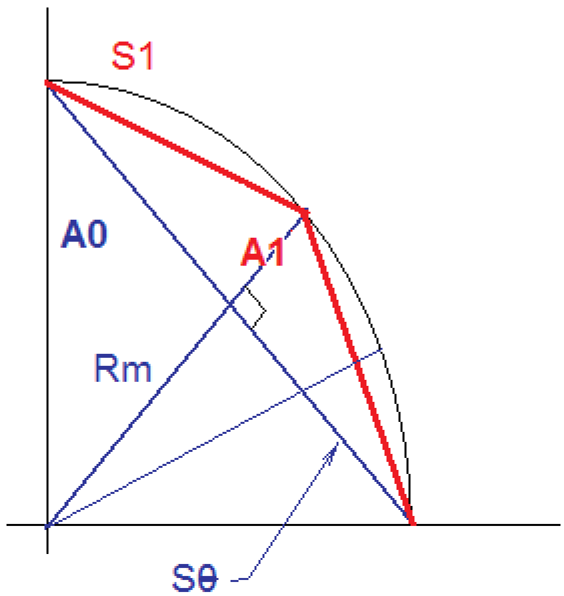
$$k\pi = \sum_{1}^{\frac{ln\infty}{ln2}} \frac{y}{2} + 2^{n-3}\left(2-\sqrt[\langle n\rangle]{2+\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)\left(\sqrt[\langle n\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)$$

$where\ \langle n\rangle\ is\ \ number\ of\ nested\ terms\ and\ (1-x)^2+(y)^2\ is\ the\ chord\ length$ for $\theta = k2\pi$

where $0 \le k \le {}^1/_2$

$$\boldsymbol{iff}\ \mathfrak{I}\ \perp\ \mathfrak{R}$$

For example area $\frac{\pi}{4}$ requires $\theta = \frac{\pi}{2}$ therefore $S_\theta = \sqrt{2}$.



$$\frac{\pi}{4} = \sum_{1}^{\frac{ln\infty}{ln2}} \frac{1}{2} + 2^{n-3}\left(2 - \sqrt[\langle n\rangle]{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{4 - 2}}}}\right)\left(\sqrt[\langle n\rangle]{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{4 - 2}}}}\right)$$

kπ from $\frac{\pi}{2}$ to π

$$k\pi = \sum_{1}^{\frac{ln\infty}{ln2}} \frac{\pi - y}{2} + 2^{n-1}\left(2 - \sqrt[\langle n\rangle]{2 + \sqrt{2 + \sqrt{2 + \cdots \sqrt{4 - {S_\theta}^2}}}}\right)\left(\sqrt[\langle n\rangle]{2 - \sqrt{2 + \sqrt{2 + \cdots \sqrt{4 - {S_\theta}^2}}}}\right)$$

where $\langle n\rangle$ is number of nested terms and $(1+x)^2 + (y)^2$ is the chord length of $k\pi$

where ${}^1\!/_2 \leq k \leq 1 \quad \boldsymbol{iff} \ \Im \ \perp \ \Re$

Note ${S_0}^2 = (1+x)^2 + (y)^2$

Proof of Convergence and Absolute Convergence of Inscribed Square Infinite Sum series

Using the ratio test or **D'Alembert's criterion**. If r exists such that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

If $r < 1$, then the series converges.

$$\lim_{n\to\infty}\frac{2^n\left(2-\sqrt[\langle n\rangle]{2+\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)\left(\sqrt[\langle n\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)}{2^{n-1}\left(2-\sqrt[\langle n-1\rangle]{2+\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)\left(\sqrt[\langle n-1\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)}$$

$$\lim_{n\to\infty}\frac{2\cdot{}^{\langle n+1\rangle}\left(2-\sqrt[\langle n\rangle]{2+\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)\left(\sqrt[\langle n\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)}{{}^{\langle n\rangle}\left(2-\sqrt[\langle n-1\rangle]{2+\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)\left(\sqrt[\langle n-1\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)}$$

$$\because\left(\sqrt[\langle n\rangle]{2-\sqrt{2+\sqrt{2+\cdots\sqrt{4-{S_\theta}^2}}}}\right)=\sin\theta\quad\text{(from ID 3a)}$$

$$=\frac{2\cdot\sin^2\frac{\theta}{2}\sin\theta}{\sin^2\theta\,\sin 2\theta}$$

As $n \to \infty; \theta \to 0$

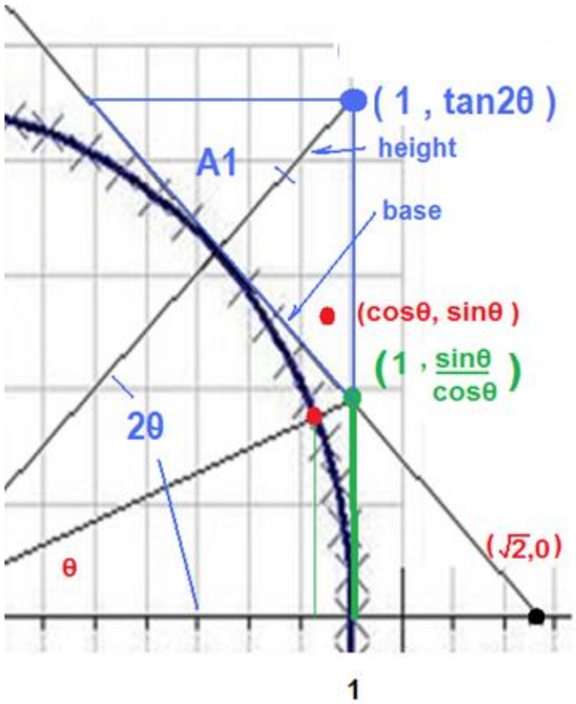
$$= \frac{2 \cdot \left(\frac{\theta}{2}\right)^2 \theta}{\theta^2 \; 2\theta} = \frac{1}{4}$$

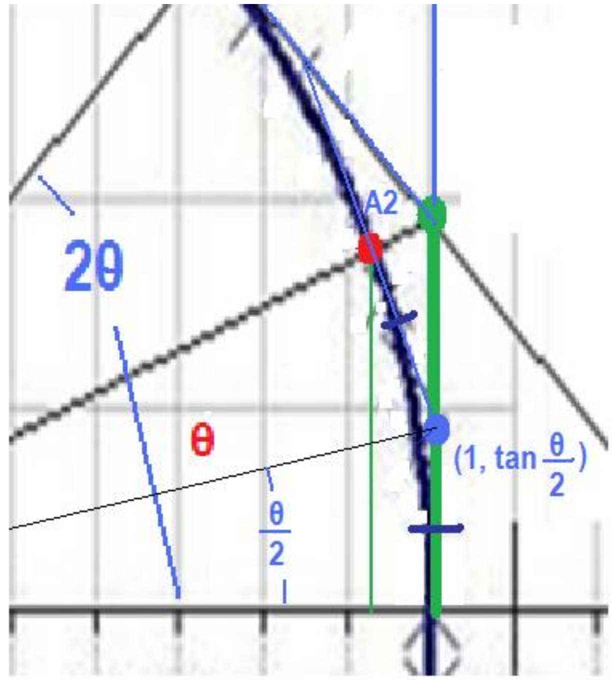
$\because r = \frac{1}{4}$ the series converges.

$\because \; r = \; \frac{1}{2} \; for \; \sum \frac{1}{2^n}$ and $\sum \frac{1}{2^n}$converges absolutely

$\therefore$ this series converges faster $\because r = \frac{1}{4}$ and therefore also must converge absolutely.

Inscribed Circle Sum series





For π take area of quadrant 1 and multiply by 4.

In quadrant 1 initial area of $1 \times 1 = 1$. By subtracting subsequently bisected fractal areas we approach area of a circle. Noting base length of triangle area is $\tan {}^{\theta}/_{2}$

$$A_1 = \left(\sqrt{1^2 + \tan^2 2\theta} - 1\right) \tan\theta$$

$$A_2 = \left(\sqrt{1^2 + \tan^2\theta} - 1\right) \cdot \tan {}^{\theta}/_{2}$$

$$\frac{\sqrt{2 - \sqrt[\langle n\rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}}{\sqrt{2 + \sqrt[\langle n\rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}} = \tan\frac{\pi}{2^{n+2}}$$

$$\sqrt{\mathbf{2} - \sqrt[\langle n\rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{\mathbf{2}}}}} = \mathbf{2}\,\mathbf{sin}\left({}^{\boldsymbol{\pi}}/_{\mathbf{2}^{n+2}}\right)$$

$$\sqrt{\mathbf{2}+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}=\mathbf{2}\cos\left(\boldsymbol{\pi}/\mathbf{2^{n+2}}\right)$$

$$A_n=\left(\frac{1-\cos\theta}{\cos\theta}\right)\tan\theta/2 \quad ; \quad where\ for\ n=1\ \ \theta=\frac{\pi}{4}$$

$$A_n=\left(\frac{1-\frac{\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}{2}}{\frac{\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}{2}}\right)\left(\frac{\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt{2+\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}\right)$$

$$A_n=\left(\frac{2-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}{\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}\right)\left(\frac{\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt{2+\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}\right)$$

$$\frac{\pi}{2}=2-\sum_{1}^{2^n=\infty}2^n\left(\frac{2-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}{\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}\right)\left(\frac{\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt{2+\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}\right)$$

$$k\pi=\frac{y}{x}-\sum_{1}^{2^n=\infty}2^n\left(\frac{2-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}}}{\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}}}\right)\left(\frac{\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-(S_0)^2}}}}}{\sqrt{2+\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-(S_0)^2}}}}}\right)$$

$for\ k\pi\ S_0\rightarrow k\pi$ For area $\frac{2\pi}{3}$ the angle of $\frac{\pi}{3}$ is the angle used for chord S_0

$$4-(S_0)^2=4-\left(\sqrt{3}\right)^2=1$$

$$\sqrt[\langle n \rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}} = \mathbf{2}\cos\left(\mathbf{k\pi}/_{\mathbf{2^n}}\right)$$

$$\frac{\sqrt{\mathbf{2}-\sqrt[\langle n \rangle]{\mathbf{1}}}}{2} = \mathbf{sin}\left(\frac{\mathbf{2\pi}}{\mathbf{3}}/_{\mathbf{2^{n+1}}}\right) \text{ for n} = 1;\ \theta = \frac{\pi}{3}$$

$$\frac{\sqrt{\mathbf{2}+\sqrt[\langle n \rangle]{\mathbf{1}}}}{2} = \cos\left(\frac{\mathbf{2\pi}}{\mathbf{3}}/_{\mathbf{2^{n+1}}}\right) = \cos\left(\frac{\boldsymbol{\pi}}{\mathbf{6}}\right)$$

Proof of Convergence and Absolute Convergence

Using the ratio test or **D'Alembert's criterion**. If r exists such that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r.$$

If r < 1, then the series converges.

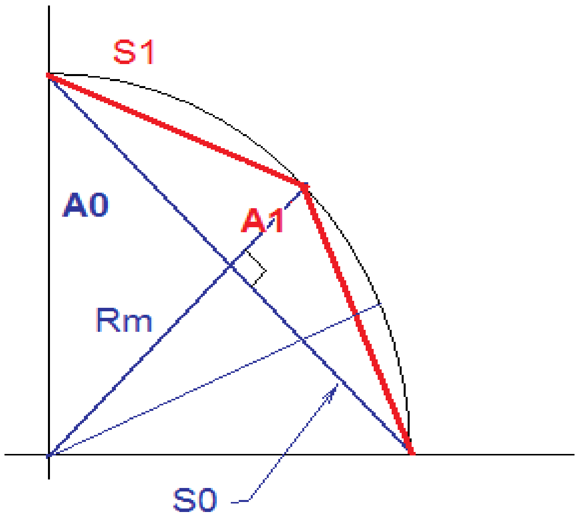
$$r = \lim_{n\to\infty}\frac{2^n\left(\frac{1-\cos\theta/_2}{\cos\theta/_2}\right)\tan\theta/_4}{2^{n-1}\left(\frac{1-\cos\theta}{\cos\theta}\right)\tan\theta/_2}$$

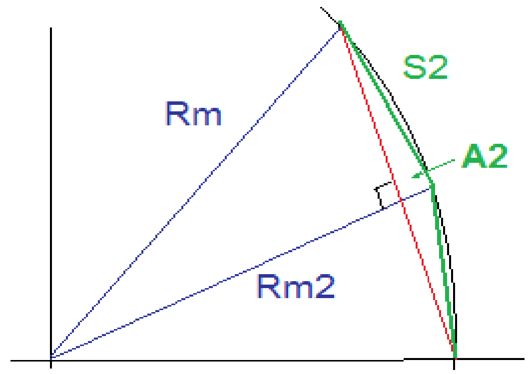
$$A_n = \left(\frac{1-\cos\theta}{\cos\theta}\right)\tan\theta/_2 = \left(\sqrt{1^2+\tan^2\theta}-1\right)\cdot\tan\theta/_2$$

$$As\ n\to\infty, \theta\to 0;\ \left(\sqrt{1^2+\tan^2\theta}-1\right)\cdot\tan\theta/_2 \to \left(\sqrt{1^2+\theta^2}-1\right)\theta/_2 \to \left(1+\theta^2/_2-1\right)\theta/_2$$

$$r = \lim_{n\to\infty}\frac{2\left(\left(\theta^2/_8\right)\theta/_4\right)}{1\left(\theta^2/_2\right)\theta/_2} = 1/_4$$

Rough work for Inscribed Square Sum series





For π take area of quadrant 1 and multiply by 4.

In general the area circumscribed by θ *between* 0 *and* π is as follows:

$$A_0 = \frac{y}{2}$$

$$A_1 = \frac{S_0}{2}(1 - R_m)$$

$$Rm = \sqrt{1^2 - \left(\frac{S_0}{2}\right)^2} = Rm = \frac{1}{2}\sqrt{4 - (S_0)^2}$$

$$A_1 = \frac{S_0}{2}\left(1 - \frac{1}{2}\sqrt{4 - (S_0)^2}\right)$$

$$= \frac{S_0}{4}\left(2 - \sqrt{4 - (S_0)^2}\right)$$

$$A_2 = \frac{S_1}{4}\left(2 - \sqrt{4 - (S_1)^2}\right)$$

New area created is

$$2A_2 = \frac{S_1}{2}\left(2 - \sqrt{4 - (S_1)^2}\right)$$

And $S_1 = \sqrt{2 - \sqrt{4 - (S_0)^2}}$ and $(S_1)^2 = 2 - \sqrt{4 - (S_0)^2}$ and $(S_0)^2 = (1 - x)^2 + (y)^2$

Rough work

$$\frac{\pi}{2} = 2 - \sum_{1}^{2^n=\infty} 2^n \left(\frac{2 - \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{\mathbf{2}}}}}{\sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{\mathbf{2}}}}} \right) \left(\sqrt{\frac{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{2}}}}} \right)$$

$$\frac{\pi}{2} = 1.570796327 = 2 - 2(0.171572875) - 4(0.016388827) - 8(0.00192956)$$

1.656854249; 1.59129894; 1.575862455

$$k\pi = \frac{2y}{x} - \sum_{1}^{2^n=\infty} 2^n \left(\frac{2 - \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{4 - (S_0)^2}}}}{\sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{4 - (S_0)^2}}}} \right) \left(\sqrt{\frac{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - (S_0)^2}}}}{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{4 - (S_0)^2}}}}} \right)$$

$$x, y \ from \ \frac{k}{2}$$

$$\frac{\pi}{3} = \frac{2y}{x} - \sum_{1}^{2^n=\infty} 2^n \left(\frac{2 - \sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{3}}}}{\sqrt[\langle n \rangle]{\mathbf{2} + \sqrt{\mathbf{2} + \cdots \sqrt{3}}}} \right) \left(\sqrt{\frac{2 - \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{3}}}}{2 + \sqrt[\langle n \rangle]{2 + \sqrt{2 + \cdots \sqrt{3}}}}} \right)$$

$$\frac{\pi}{3} = 1.047197551 = \frac{2}{\sqrt{3}} - 2(0.041451884) - 4(0.004644197) - 8(0.00056923)$$

1.07179677; 1.053219981;1.048666134

$for\ k\pi\ S_0 \to k\pi$ For area $\frac{\pi}{3}$ the angle of $\frac{\pi}{3}$ is the angle used for chord S_0

$$4 - (S_0)^2 = 4 - \left(\sqrt{1}\right)^2 = 3$$

$$\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}}=\mathbf{2}\cos\left(\mathbf{k\pi}/_{\mathbf{2^n}}\right)$$

$$\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-(S_0)^2}}}}=\mathbf{2}\sin\left(\mathbf{k\pi}/_{\mathbf{2^{n+1}}}\right)$$

$$\frac{\sqrt{2+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{4-(S_0)^2}}}}}{\sqrt{2-\sqrt[\langle n\rangle]{2+\sqrt{2+\cdots\sqrt{4-(S_0)^2}}}}}=\tan\left(\mathbf{k\pi}/_{\mathbf{2^{n+1}}}\right)$$

Other observations and identities.

$$\lim_{2^n\to\infty}\mathbf{2^n}\tan{}^{\boldsymbol{\pi}}/_{\mathbf{2^n}}=\boldsymbol{\pi}$$

$$\lim_{2^n\to\infty}\mathbf{2^n}\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}=\frac{\pi}{2};\ \textit{where } \langle n\rangle \textit{ is number of nested terms}$$

And further yielding the trig identity $\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}=\mathbf{2}\sin\left(^{\boldsymbol{\pi}}/_{\mathbf{2^{n+2}}}\right)$

$$\sqrt{\mathbf{2}+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}=\mathbf{2}\cos\left(^{\boldsymbol{\pi}}/_{\mathbf{2^{n+2}}}\right)$$

$$\frac{\sqrt{\mathbf{2}-\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}}{\sqrt{\mathbf{2}+\sqrt[\langle n\rangle]{\mathbf{2}+\sqrt{\mathbf{2}+\cdots\sqrt{\mathbf{2}}}}}}=\tan\left(^{\boldsymbol{\pi}}/_{\mathbf{2^{n+2}}}\right)$$

By putting the circle inside the square and chipping away at it by recursively doubling and using smaller tangents we arrive at $\lim_{2^n \to \infty} 2^{n+2} \frac{\sqrt{2 - \sqrt[\langle n \rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}}{\sqrt{2 + \sqrt[\langle n \rangle]{2+\sqrt{2+\cdots\sqrt{2}}}}} = \pi$

$$\lim_{2^n \to \infty} 2^{n+1} \tan {}^{\pi}/_{2^{n+1}} = \pi$$

$$2^n 2 \sin\left({}^{2\pi}/_{2^n}\right)$$