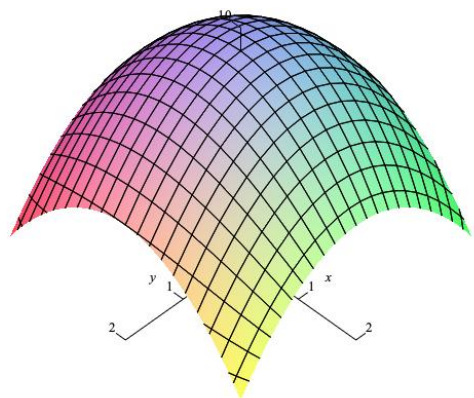
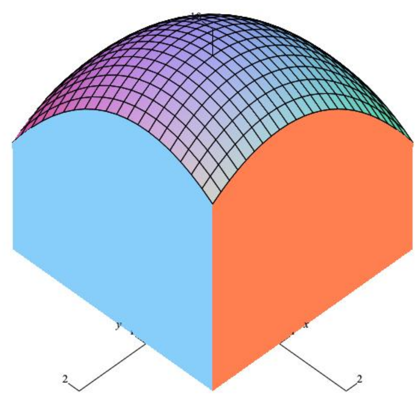
Calculus 3 - Directional Derivative

When we first introduced partial derivatives, we took slices of the surface

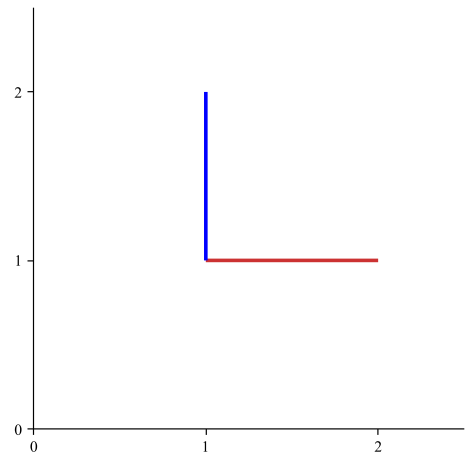




and defined the two derivatives

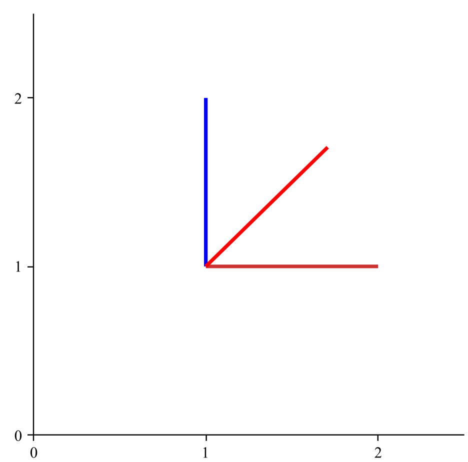
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \tag{1a}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}. \tag{1b}$$



A view from the top, we see for each slice we moved in the x direction or y direction and follow the vectors $< 1, 0 >$ or $< 0, 1 >$.
We now want to ask, suppose we wish to move in another direction, say in the direction of the unit vector

$$\vec{u} = < u_1, u_2 > \tag{2}$$



Can we calculate the derivative if we move in this direction. The term *directional* derivative is used here.

Using the derivative like in calc 1 where

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3}$$

we use

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(x + u_1 h, y + u_2 h) - f(x, y)}{h}. \tag{4a}$$

To get an idea on how to calculate this (not the long way) we will first fix (x, y) say to (a, b) so

$$D_{\vec{u}} f = \lim_{h \to 0} \frac{f(a + u_1 h, b + u_2 h) - f(a, b)}{h}. \tag{5a}$$

We define

$$g(h) = f(a + u_1 h, b + u_2 h) \tag{6}$$

so (5) becomes

$$\lim_{h \to 0} \frac{g(h) - g(0)}{h} \tag{7}$$

which from calc 1 is $g'(0)$. So we calculate $g'(h)$ from (6) then substitute in $h = 0$. From the last class, we us a type 1 chain rule so

$$g'(h) = f_1 u_1 + f_2 u_2 \tag{8}$$

where the subscripts in f refer to differentiation with respect to that argument. Now when $h = 0$ we obtain

$$g'(0) = f_x u_1 + f_y u_2 \tag{9}$$

and thus we obtain the directional derivative

$$D_{\vec{u}}f = f_x(a,b)u_1 + f_y(a,b)u_2. \tag{10}$$

Note: When we have $< 1,0 >$ and $< 0,1 >$, we obtain the usual x and y derivatives. Now it usual to rewrite (10) as a dot product of two vectors so

$$D_{\vec{u}}f = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle \tag{11a}$$

Gradient Vector

At this point we wish to define a new vector called the *gradient* vector. It is defined as

$$\nabla f = \langle f_x(x,y), f_y(x,y)\rangle \tag{12}$$

and so the directional derivative is given as

$$D_{\vec{u}}f = \nabla f|_P \cdot \vec{u}. \tag{13}$$

Let us look at some examples.

Example 1 Pg 928, #8

Find the directional derivative of the function at P in the direction given

$$f = x^3 - y^3, \quad P(4,3), \quad \vec{v} =< 1,1 > . \tag{14}$$

Soln.

First we find the unit vector. The magnitude of $\vec{v} = \sqrt{2}$ or the unit vector is

$\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$. The gradient is $\nabla f =< 3x^2, -3y^2 >$. Thus, the directional derivative is

$$D_{\vec{u}} f = \nabla f|_P \cdot \vec{u} = \langle 48, -27 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \frac{21}{\sqrt{2}} \tag{15}$$

Example 2 Pg 928, #12

Find the directional derivative of the function at P in the direction of $\overrightarrow{PQ}$

$$f = \cos(x - y), \quad P(0, \pi), \quad Q\left(\frac{\pi}{2}, 0\right) \tag{16}$$

Soln.

First we find the unit vector. The vector we follow $\overrightarrow{PQ} = \langle \frac{\pi}{2}, -\pi \rangle$. The magnitude of this is $\frac{\sqrt{5}\pi}{2}$. Dividing by the magnitude gives $\vec{u} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$. Next, calculate the gradient. The gradient is

$$\nabla f = \langle -\sin(x - y), \sin(x - y) \rangle.$$

At the point P this becomes

$$\nabla f|_P = \langle -\sin(-\pi), \sin(-\pi) \rangle = \langle 0, 0 \rangle.$$

Thus, the directional derivative is

$$D_{\vec{u}} f = \nabla f|_P \cdot \vec{u} = \langle 0, 0 \rangle \cdot \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle = 0 \tag{17}$$

Maximum Increase/Decrease

Consider the following problem. Find the directional derivative of

$$z = 2 - x^2 - y^2$$

at the point $P(1,1)$ when we follow the vectors

$$\langle 1,0\rangle, \quad \langle 0,1\rangle, \quad \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle.$$

The gradient is

$$\nabla f = \langle -2x, -2y\rangle. \tag{18}$$

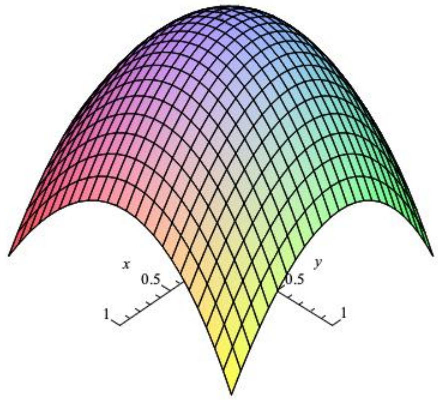
Then

$$\nabla f|_P = \langle -2, -2\rangle. \tag{19}$$

The three directional derivatives are

$$\begin{aligned} &(1) \quad D_{\vec{u}} f = \langle -2,-2\rangle \cdot \langle 1,0\rangle = -2, \\ &(2) \quad D_{\vec{u}} f = \langle -2,-2\rangle \cdot \langle 0,1\rangle = -2, \\ &(3) \quad D_{\vec{u}} f = \langle -2,-2\rangle \cdot \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle = -2\sqrt{2}. \end{aligned} \tag{20}$$

so we see following the third vector, the decrease (negative slope) is larger than following the other two vectors. So we ask, in what direction should we move to find the maximum/minimum increase.



Let us return to our definition (13)

$$\begin{aligned} D_{\vec{u}} f &= \nabla f|_P \cdot \vec{u} \\ &= \|\nabla f|_P\| \|\vec{u}\| \cos\theta \end{aligned} \tag{21}$$

Now $\cos\theta$ will vary from -1 to 1 with it's maximum and minimum being at $\theta = 0$ and $\theta = \pi$. So for maximum/minimum increase follow the direction of the gradient.

3D Gradients

Gradients easily extend to function of more variables. For example if

$$f(x, y, z) = x^2 + 3y + e^z \tag{22}$$

then

$$\begin{aligned} \nabla f &= \langle f_x, f_y, f_z \rangle \\ &= \langle 2x, 3, e^z \rangle. \end{aligned} \tag{23}$$