Deploying Robots With Two Sensors in $K_{1,6}$-Free Graphs

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Abstract: Let G be a graph of minimum degree at least 2 with no induced subgraph isomorphic to $K_{1,6}$. We prove that if G is not isomorphic to one of eight exceptional graphs, then it is possible to assign two-element subsets of $\{1, 2, 3, 4, 5\}$ to the vertices of G in such a way that for every $i \in \{1, 2, 3, 4, 5\}$ and every vertex $v \in V(G)$ the label i is assigned to v or

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one of its neighbors. It follows that G has fractional domatic number at least 5/2. This is motivated by a problem in robotics and generalizes a result of Fujita, Yamashita, and Kameda who proved that the same conclusion holds for all 3-regular graphs. J. Graph Theory 00: 1–17, 2015

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1. INTRODUCTION

The problem under consideration in this article is motivated by a problem encountered both in the multiagent robotics and mobile sensor networks domains. Common to both of these two application areas is a collection of agents that are equipped with sensors of various types, used for tasks such as environmental modeling, exploration of unknown terrains, surveillance of remote locations, and the establishment of sensor coverage for the purpose of event detection. Due to the scale of the multirobot network, the agents have to act based on locally available information, and under various such distributed coordinated schemes, for example, [1], the robots interact and communicate with each other in order to gain the information needed to make informed decisions. These interactions, in turn, define an information exchange network that allows us to model the agents as vertices and information exchange channels as edges in a graph. The interagent interactions moreover allow the agents to complement each others' resources and capabilities; thus enhancing the collective functionality of the system. As a result, the underlying network topology of multirobot networks plays a crucial role in achieving the system level objectives within the network in a distributed manner.

As an example, consider an application in which a group of robots is deployed at some remote location for the purpose of environmental monitoring. Each robot needs to obtain information about s different sensing modalities (e.g., temperature, humidity, barometric pressure, and so on). However, owing to certain constraints such as power limitations and hardware footprints, an individual robot can have a maximum of $r < s$ sensors installed on it. As a result, the robots need to collect data concerning the remaining $s - r$ sensing modalities from neighboring robots through the information exchange network. In other words, for every robot v and every type of sensor, either v or one of its neighboring robots must carry a sensor of that type.

As already stated, the multirobot network can be modeled as a graph G, in which the vertex set represents robots, and the edges correspond to the interactions among robots. Typically, a robot may transmit data to other robots lying within a certain Euclidean distance, say R, away from it. Thus, an edge is formed between nodes v and u whenever $\|v - u\| \leq R$. This results in an *R-disk proximity graph* model of the network, which is the typical model employed when studying multirobot networks. As such, any graph class under consideration must be rich enough to capture this model for it to be relevant to robotics. In such a graph, a disk of radius R, which represents the transmission or interaction range of the node, is associated with every node v that lies at the center of the disk. An edge exists between v and all such nodes that lie within the disk of u. R-disk graphs are one of the most frequently used models for the analysis of the network topology related aspects of multirobot systems, wireless sensor networks, and other ad hoc networks (e.g., see [5]). R-disk graphs are geometric graphs as the existence of edges between vertices depends on the geometric configuration of vertices. However, the geometric property of such graphs can be translated into a graph-theoretic one. In fact, it can be shown that R-disk graphs are indeed $K_{1,6}$-free, and this key observation motivates the study of $K_{1,6}$-free graphs in multiagent robotics.

In this article, we study what is the maximum number of sensors that can be accommodated in a multirobot network if each robot can have at most two types of sensors. Our main result states that under some mild conditions, it is possible to assign two distinct labels to each vertex in a $K_{1,6}$-free graph such that a set of five distinct labels always exist in the closed neighborhood of every vertex in G.

The same problem arises in various situations of locating facilities in a network. Let us assume that every vertex of a graph can access only resources located at neighboring vertices or at the vertex itself. Now if some resource (such as a file, a printer, or other service) must be accessible from every vertex of the graph, then copies of that resource need to be distributed over the network to form a "dominating set." If every vertex of the graph has the capacity to accommodate at most r distinct resources, then asking for the maximum number of resources that can be made available to every vertex of the graph leads to the same mathematical question as the problem of the previous paragraph.

Let us be more precise now. By a *graph* we mean a finite, simple, undirected graph; that is, loops and parallel edges are not allowed. For a vertex v of a graph G, we denote the set of neighbors of v by $N(v)$, and define $N[v]$, the *closed neighborhood of* v, to be $N(v) \cup \{v\}$. Let $r \geq 1$ be an integer. Let f be a function that maps the vertices of G to r-element subsets of some set X. We define $R(f)$ to be the union of $f(v)$ over all vertices v of G. Following [4], we say that f is an *r-configuration on G* if for every $x \in R(f)$ and every vertex $v \in V(G)$ we have $x \in f(u)$ for some $u \in N[v]$. We define $D_r(G)$ to be the maximum of $|R(f)|$ over all r-configurations on G. Thus, given a graph G and integer $r \geq 1$ the problems of the previous two paragraphs ask for the value of $D_r(G)$.

The parameter $D_1(G)$ is known in the literature as the *domatic number* of G. It was introduced by Cockayne and Hedetniemi [2] and has since then been the subject of a large number of publications. Obviously $D_1(G)$ is at most the minimum degree of G plus one, but testing whether $D_1(G) \geq k$ is NP-complete for all $k \geq 3$. (Testing $D_1(G) \geq 2$ is easy, because $D_1(G) \geq 2$ if and only if G has no isolated vertex.) A $(1 + o(1)) \ln n$-approximation algorithm for $D_1(G)$ was found by Feige, Halldórsson, Kortsarz, and Srinivasan [3], who also showed that their approximation factor is essentially best possible.

Fujita, Yamashita, and Kameda proved in [4] that $D_2(G) \geq 5$ for all 3-regular graphs. The purpose of this article is to generalize their result to a larger class of graphs, as follows. We denote the cycle on n vertices by C_n. By $C_4 \cdot C_4$ we mean the graph obtained from two disjoint cycles on four vertices by identifying a vertex in the first cycle with a vertex in the second cycle. We denote by $G_1, G_2, G_3,$ and G_4 the graphs shown in Figure 1.

Theorem 1. *Let G be a graph of minimum degree at least 2 with no induced subgraph isomorphic to $K_{1,6}$. If no component of G is isomorphic to a member of $\{C_4, C_7, C_4 \cdot C_4, K_{2,3}, G_i : 1 \leq i \leq 4\}$, then $D_2(G) \geq 5$.*

As stated earlier, the generalization to $K_{1,6}$-free graphs is of interest in multiagent robotics, because the class of $K_{1,6}$-free graphs includes the class of R-disk graphs. For the sake of brevity, let us define a *configuration* on a graph G to mean a 2-configuration f with $R(f) = \{1, 2, 3, 4, 5\}$. Thus, the conclusion of Theorem 1 is equivalent to saying that G

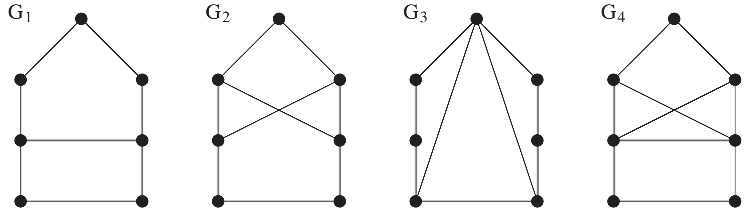

FIGURE 1. Graphs G_1, G_2, G_3, and G_4.

has a configuration. Our proof is algorithmic and gives a polynomial-time algorithm to find a configuration. We say that a graph G is *configurable* if it admits a configuration. Theorem 1 has the following two corollaries.

Corollary 2. *If G is a connected graph of minimum degree at least 2 with no induced subgraph isomorphic to $K_{1,6}$, and G is not isomorphic to a member of $\{C_4, C_7, C_4 \cdot C_4, K_{2,3}, G_i : 1 \leq i \leq 4\}$, then for any positive integer r, $D_r(G) \geq \lfloor 5r/2 \rfloor$.*

Proof. Since G has no isolated vertex, we have $D_1(G) \geq 1$. Thus G has a 1-configuration h with $R(h) = \{1, 2\}$. By Theorem 1 the graph G has a configuration, say f. For $v \in V(G)$ we define $g(v)$ to be the set of all pairs (i, j), where $i \in f(v)$ and $j \in \{1, 2, \ldots, \lfloor r/2 \rfloor\}$, and let $g'(v) := g(v) \cup h(v)$. If r is even, then g is an r-configuration with $|R(g)| = 5r/2$, and if r is odd, then g' is an r-configuration with $|R(g')| = 5(r-1)/2 + 2 = \lfloor 5r/2 \rfloor$, as desired. □

In the context of R-disk graphs, which are widely used to model inter-communication and information exchange among nodes in multirobot and wireless sensor networks, we can restate the above result using the fact that R-disk graphs are always $K_{1,6}$-free, and can never be isomorphic to $K_{2,3}$, as shown in [7].

Corollary 3. *If G is a connected R-disk graph of minimum degree at least 2, and G is not isomorphic to a member of $\{C_4, C_7, C_4 \cdot C_4, K_{2,3}, G_i : 1 \leq i \leq 4\}$, then for any positive integer r, $D_r(G) \geq \lfloor 5r/2 \rfloor$.*

The *fractional domatic number* of a graph G, introduced in [6], is the supremum of a/b such that G has a b-configuration f with $|R(f)| = a$. This is the optimum of the LP relaxation of the domatic number problem, and that justifies the name. It follows that the supremum is attained. Theorem 1 implies that every graph that satisfies the hypotheses of the theorem has fractional domatic number at least $5/2$.

The article is organized as follows. In Section 2, we prove some lemmas about extending a configuration from a subgraph of a graph. In section 3, we prove the main theorem under the additional hypothesis that no two vertices of degree at least 3 are adjacent. In section 4 we prove the main theorem and give two examples that show limitations to possible extensions.

2. PRELIMINARY LEMMAS

An (α, β)-*star* is the graph obtained by identifying one end of each of α paths of length 1 and β paths of length 2. In other words, the vertex-set may be labeled $\{w, x_i, y_j, z_j : 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$ so that the edge-set is $\{wx_i, wy_j, y_jz_j : 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$. Note that an $(\alpha, 0)$-star is isomorphic to $K_{1,\alpha}$. We denote by $[5]^2$ the set of all two-element subsets of $\{1, 2, 3, 4, 5\}$. If G is a graph, $f : V(G) \to [5]^2$, and $v \in V(G)$, then we say that v *is satisfied* with respect to f if $\bigcup_{u \in N[v]} f(u) = \{1, 2, 3, 4, 5\}$. When there is no danger of confusion we will omit the reference to f.

Lemma 4. *Let $v_1v_2v_3v_4$ be a path of length 3, and $f : \{v_1, v_4\} \to [5]^2$ with $f(v_1) \cap f(v_4)$ nonempty. If $a, b \in \{1, 2, 3, 4, 5\} \setminus f(v_1)$, then f can be extended to $\{v_1, v_2, v_3, v_4\}$ in such a way that v_2 and v_3 are satisfied and $f(v_2) = \{a, b\}$.*

Proof. Without loss of generality, $f(v_1) = \{1, 2\}$, $1 \in f(v_4)$, and $f(v_2) = \{a, b\} = \{3, 4\}$. Then, setting $f(v_3) = \{2, 5\}$ completes the proof. □

Lemma 5. *Let H and S be disjoint subgraphs of a graph G, and let $\alpha, \beta \geq 0$ be integers such that either $\alpha + 3\beta \leq 9$ or $(\alpha, \beta) = (1, 3)$. Let H be configurable and let S be either a path of length at least two or an (α, β)-star. If every vertex of S of degree 1 is adjacent to some vertex of H, then the subgraph of G induced by $V(H) \cup V(S)$ is configurable.*

Proof. Let f be a configuration on H. First, suppose that $S = v_1v_2...v_k$ is a path of length at least 2 (so $k \geq 3$), and that the ends of S are adjacent to vertices x and y of H. Note that x and y may be the same vertex. There are three cases depending on the cardinality of $f(x) \cap f(y)$ and three cases depending on the residue of k modulo three. Without loss of generality we may assume that $f(x) = f(y) = \{1, 2\}$, or $f(x) = \{1, 2\}$ and $f(y) = \{1, 3\}$, or $f(x) = \{1, 2\}$ and $f(y) = \{3, 4\}$. Then f can be extended to $V(H) \cup V(S)$ according to the following table, where t runs from 1 through $\lfloor k/3 \rfloor - 1$.

k (mod 3)	$f(x)$	$f(v_{3t+1})$	$f(v_{3t+2})$	$f(v_{3t+3})$	$f(v_{k-1})$	$f(v_k)$	$f(y)$
0	{1, 2}	{1,3}	{4,5}	{2,3}	x	x	{1,2}
0	{1, 2}	{3,4}	{1,5}	{2,4}	x	x	{1,3}
0	{1, 2}	{3,4}	{1,5}	{1,2}	x	x	{3,4}
1	{1, 2}	{3,4}	{1,5}	{2,5}	x	{3,4}	{1,2}
1	{1, 2}	{3,4}	{1,5}	{2,5}	x	{3,4}	{1,3}
1	{1, 2}	{3,5}	{1,4}	{1,2}	x	{3,5}	{3,4}
2	{1, 2}	{3,4}	{1,5}	{1,2}	{3,4}	{1,5}	{1,2}
2	{1, 2}	{3,4}	{2,5}	{1,2}	{3,4}	{2,5}	{1,3}
2	{1, 2}	{3,4}	{1,5}	{2,4}	{1,3}	{2,5}	{3,4}

Now we assume that S is a (α, β)-star, where $\alpha + \beta \geq 3$, $\alpha + 3\beta \leq 9$, or $(\alpha, \beta) = (1, 3)$. Let $V(S) = \{w, x_i, y_j, z_j : 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$, $E(S) = \{wx_i, wy_j, y_jz_j : 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$, and x_i is adjacent to u_i, where u_i is in H, for all $1 \leq i \leq \alpha$, and z_j is adjacent to v_j, where v_j is in H, for all $1 \leq i \leq \beta$.

We say that u_i *forbids* the set $f(u_i)$ and that v_j *forbids* the three 2-element subsets of $[5] - f(v_j)$. We claim that there is an element of $[5]^2$ that is not forbidden by any u_i or v_j. Indeed, this is clear if $\alpha + 3\beta \leq 9$. But if $\beta = 3$, then the vertices v_1, v_2, and v_3 collectively forbid at most eight sets, and hence the claim holds even when $\alpha = 1$ and $\beta = 3$. We define $f(w)$ to be an element of $[5]^2$ that is not forbidden by any u_i or v_j. Furthermore, if $\beta = 0$ and $|\bigcup_{i=1}^{\alpha} f(u_i)| \leq 3$, then we choose $f(w)$ disjoint from every $f(u_i)$.

If $\beta \geq 1$, then we choose $f(x_i)$, $f(y_j)$, and $f(z_j)$ for $i = 1, 2, \ldots, \alpha$ and $j = 1, 2, \ldots, \beta - 1$ in such a way that the vertices x_i, y_j, and z_j are satisfied. Then w sees at least three values under f since any neighbor of w already assigned a value does not have the exact same assignment as w. So by Lemma 4 applied to the path $wy_\beta z_\beta v_\beta$ we can assign $f(y_\beta)$ and $f(z_\beta)$ in such a way that y_β, z_β, and w are satisfied. This completes the case $\beta \geq 1$.

So we may assume $\beta = 0$. We assign $f(x_i)$ for $i = 1, 2, \ldots, \alpha$ such that x_i is satisfied, $f(x_i) \cap f(w) = \emptyset$, and, if possible, not all $f(x_i)$ are the same. Then w is satisfied, unless the sets $f(x_i)$ are all equal, and so from the symmetry we may assume that $f(w) = \{1, 2\}$ and $f(x_i) = \{3, 4\}$ for all $i = 1, 2, \ldots, \alpha$. But then the choice of $f(x_i)$ implies that $f(u_i) \subseteq \{1, 2, 5\}$, contrary to the choice of $f(w)$. □

Lemma 6. *Let G be a graph, and let $P = xv_1v_2v_3y$ be a path in G. If x is adjacent to y, then let $H := G\backslash\{v_1, v_2, v_3\}$; otherwise let H be the graph obtained from $G\backslash\{v_1, v_2, v_3\}$ by adding the edge xy. If H is configurable, then G is configurable.*

Proof. Let f be a configuration on H. We shall extend f to $V(G)$. If $f(x) = f(y)$, say $f(x) = \{1, 2\}$, then $H \setminus xy$ is also configurable, so we can extend f to $V(G)$ by Lemma 5. So we may assume that $f(x) \neq f(y)$; that is, $|f(x) \cup f(y)| \geq 3$. Define $g : V(G) \to [5]^2$ by $g(v_1) = f(y)$, $g(v_3) = f(x)$, let $g(v_2)$ be a 2-element subset of $[5]^2$ containing $\{1, 2, 3, 4, 5\} \setminus (f(x) \cup f(y))$, and let $g(v) = f(v)$ for all $v \in V(G) \setminus \{v_1, v_2, v_3\}$. Then it is clear that g is a configuration on G. □

Let G be a graph and v a vertex of G. Let f be a function mapping $V(G)$ to $[5]^2$ and $c \in [5]$. Then we say that v is *missing* c if $c \notin \bigcup_{u \in N[v]} f(u)$.

Lemma 7. *Let H be C_4, C_7, or a configurable graph, and let u_0 be a vertex of H. Let G be a graph, where $V(G) = V(H) \cup \{u_i, w_j : 1 \leq i \leq k, 1 \leq j \leq m\}$ and $E(G) = E(H) \cup \{u_iu_{i+1}, u_kw_1, w_jw_{j+1}, w_mw_1 : 0 \leq i \leq k-1, 1 \leq j \leq m-1\}$ for some nonnegative integer k and integer m with $m \geq 3$. Then G is configurable.*

Proof. By Lemma 6 we may assume that $k = 0, 1,$ or 2. Let C be the cycle $w_1w_2...w_mw_1$. Since H is C_4, C_7, or a configurable graph, we may satisfy every vertex of H except possibly u_0 and u_0 is missing at most two colors. So we may assume $f(u_0) = \{1, 2\}$ and that u_0 is missing 3 and 4. Similarly we may choose f on C in such a way that every vertex of C except possibly w_1 is satisfied, and that w_1 is missing at most two colors.

If $k = 0$ we choose f on C so that $f(w_1) = \{3, 4\}$ and the colors missing at w_1 are 1 and 2. If $k = 1$, we choose f on C so that $f(w_1) = \{2, 5\}$ and the colors missing at w_1 are 3 and 4. We set $f(u_1) = \{3, 4\}$. Finally, if $k = 2$, we choose f on C so that $f(w_1) = \{2, 3\}$ and the colors missing at w_1 are 1 and 5. We set $f(u_1) = \{3, 4\}$ and $f(u_2) = \{1, 5\}$. □

Lemma 8. *Let H be a configurable graph, and let f be a configuration on G. If G is obtained from H by either*

- *adding a vertex v and two edges vx and vy to H, where x, y are vertices of H and $f(x) \neq f(y)$, or*
- *adding two vertices u and v and three edges xu, uv, and vy to H, where xand y are vertices of H and $f(x) \cap f(y) \neq \emptyset$,*

then f can be extended to G.

Proof. This is easy to verify. □

A graph G is said to be obtained from a graph H by *attaching* a path P if G is obtained from the disjoint union of H and P by adding two edges v_1x and v_ky, where v_1 and v_k are the ends of P, and x and y are vertices of H. A graph G is said to be obtained from a graph H by *adding* a path P if G is obtained from the disjoint union of H and P by identifying one end of P and x and identifying the other end of P and y, where x and y are distinct vertices of H.

Lemma 9. *Let C be a cycle of length of 5 or 6. If G is obtained from C by adding a path of length 2 or 3 between two nonadjacent vertices in C, then G is configurable.*

Proof. Let $C = v_1v_2...v_kv_1$, and P be the path in $G \setminus C$ where the end of P is adjacent to vertices u and v of C in G. If C is C_5, then we define a function $f : V(C) \to [5]^2$ by $f(v_i) = \{i, i+3\}$ for each $i = 1, 2, 3, 4, 5$, where the addition is modulo five. If C is C_6, then define $f(v_1) = \{1, 3\}, f(v_2) = \{2, 4\}, f(v_3) = \{1, 5\}, f(v_4) = \{2, 3\}, f(v_5) = \{1, 4\}, f(v_6) = \{2, 5\}$. So $f(x) \neq f(y)$ for all distinct vertices x and y in C, and $f(x) \cap f(y) \neq \emptyset$ for all nonadjacent two vertices x and y in C. Hence f can be extended to G by Lemma 8 since P is a path on one or two vertices. □

Lemma 10. *Let x and y be vertices of a configurable graph H, let $C = v_1v_2...v_5v_1$ be a cycle of length 5, and let $P = u_1u_2...u_p$ and $Q = w_1w_2...w_q$ be paths, where $p, q \in \{1, 2\}$. Assume that H, C, P, and Q are pairwise disjoint. If G is the graph with $V(G) = V(H) \cup V(C) \cup V(P) \cup V(Q)$ and $E(G) = E(H) \cup E(C) \cup E(P) \cup E(Q) \cup \{xu_1, u_pv_1, yw_1, w_qv_3\}$, then G is configurable.*

Proof. Let f be a configuration on H. We shall extend f to G. If $f(x) \cap f(y)$ is nonempty, say $1 \in f(x) \cap f(y)$, then let a and b are two distinct numbers in $\{1, 2, 3, 4, 5\} \setminus (f(x) \cup f(y))$, and define $f(v_1) = \{1, a\}$ and $f(v_3) = \{1, b\}$. If $f(x)$ is disjoint from $f(y)$, say $f(x) = \{1, 2\}$ and $f(y) = \{3, 4\}$, then define $f(v_1) = \{1, 3\}$ and $f(v_3) = \{1, 4\}$. Without loss of generality, we may assume that $a = 3$ and $b = 4$. Then we further define $f(v_2) = \{2, 5\}, f(v_4) = \{3, 5\}$, and $f(v_5) = \{2, 4\}$ so that every vertex of C is satisfied. By Lemma 8, there is a way to define f on $V(P) \cup V(Q)$ such that f is a configuration on G. □

Let us recall that the graph $C_4 \cdot C_4$ was defined in Section 1.

Lemma 11. *Let G be a graph obtained by attaching a path $P = v_1v_2...v_k$ to a cycle C with v_1 adjacent to x and v_k adjacent to y, for some vertices x and y in C, where $k \geq 3$. If G is not isomorphic to $C_4 \cdot C_4$ or G_1, then G is configurable.*

Proof. If x is adjacent to y in C, then G is a cycle with a chord. So G is configurable when the cycle has length not 4 or 7. It is easy to check that G is configurable when the cycle has length 4. And since G is not isomorphic to G_1, G is also configurable when the cycle has length 7 by Lemma 9. So we may assume that x is not adjacent to y in C. In other words, either x equals y, or x and y are nonadjacent.

If the length of C is not 4 or 7, then this lemma follows directly from Lemma 5. So we may assume that the length of $C = u_1u_2...u_{|C|}u_1$ is 4 or 7. Also, we may assume that $3 \leq k \leq 5$ by Lemma 6. Without loss of generality, we assume that $x = u_1$.

Case 1: $C = C_4$ and $x = y$. Then $k = 4$ or 5 since G is not isomorphic to $C_4 \cdot C_4$. So G is isomorphic to the graph obtained by attaching a path of order 3 to C_5 or C_6, and hence G is configurable by Lemma 5.

Case 2: $C = C_4$ and $x \neq y$. We may assume that $y = u_3$. If $k = 3$ or 5, then $u_1v_1v_2...v_ku_3u_2u_1$ is a cycle of length 6 or 8, so it is configurable, and there is a configuration f on it. Then we can extend f to G by assigning that $f(u_3) = f(u_1)$, so G is configurable. If $k = 4$, then we define a configuration on G by $f(u_1) = \{1, 2\}, f(u_2) = \{3, 5\}, f(u_3) = \{3, 4\}, f(u_4) = \{2, 5\}, f(v_1) = \{1, 4\}, f(v_2) = \{3, 5\}, f(v_3) = \{2, 5\}, and f(v_4) = \{1, 4\}$.

Case 3: $C = C_7$ and $x = y$. We may assume that $x = y = u_1$. If $k = 4$ or 5, then G is isomorphic to the graph obtained by attaching a path of order 6 to C_5 or C_6, so G is configurable by Lemma 5. If $k = 3$, then we can define a configuration on G by $f(u_1) = \{1, 2\}, f(u_2) = \{3, 4\}, f(u_3) = \{1, 5\}, f(u_4) = \{2, 3\}, f(u_5) = \{1, 4\}, f(u_6) = \{2, 5\}, f(u_7) = \{3, 4\}, f(v_1) = \{1, 5\}, f(v_2) = \{3, 4\}, and f(v_3) = \{2, 5\}$.

Case 4: $C = C_7, x = u_1$, and $y = u_6$. If $k = 3$ or 5, then G is isomorphic to the graph obtained by attaching a path of order 4 to C_6 or C_8, so G is configurable by Lemma 5. If $k = 4$, then we can define a configuration on G by $f(u_1) = \{1, 2\}, f(u_2) = \{3, 4\}, f(u_3) = \{3, 5\}, f(u_4) = \{1, 2\}, f(u_5) = \{4, 5\}, f(u_6) = \{3, 4\}, f(u_7) = \{3, 5\}, f(v_1) = \{1, 5\}, f(v_2) = \{3, 4\}, f(v_3) = \{2, 5\}, and f(v_4) = \{1, 2\}$.

Case 5: $C = C_7$, $x = u_1$, and $y = u_5$. If $k = 4$ or 5, then G is isomorphic to the graph obtained by attaching a path of order 3 to C_8 or C_9, so G is configurable by Lemma 5. If $k = 4$, then we can define a configuration on G by $f(u_1) = \{1, 2\}, f(u_2) = \{1, 3\}, f(u_3) = \{4, 5\}, f(u_4) = \{2, 3\}, f(u_5) = \{1, 2\}, f(u_6) = \{4, 5\}, f(u_7) = \{3, 4\}, f(v_1) = \{1, 5\}, f(v_2) = \{3, 4\}$, and$f(v_3) = \{2, 5\}$. □

Lemma 12. *The graph $K_{2,4}$ is configurable.*

Proof. Let $V(K_{2,4}) = \{x_1, x_2, y_1, y_2, y_3, y_4\}$, $E(K_{2,4}) = \{x_iy_j : 1 \leq i \leq 2, 1 \leq j \leq 4\}$. We define a configuration on $K_{2,4}$ by $f(x_1) = \{1, 2\}, f(x_2) = \{3, 4\}, f(y_1) = \{3, 5\}, f(y_2) = \{4, 5\}, f(y_3) = \{1, 5\}, and f(y_4) = \{2, 5\}$. □

Lemma 13. *If a graph G is obtained from $C_4 \cdot C_4$ or $K_{2,3}$ by attaching a path, then G is configurable.*

Proof. First, we assume that G obtained from $C_4 \cdot C_4$ by attaching a path $v_1v_2...v_k$, where v_1 is adjacent to x and v_k is adjacent to y for some vertices x and y in $C_4 \cdot C_4$. We write the vertex set of $C_4 \cdot C_4$ as $\{u_1, u_2, u_3, v, w_1, w_2, w_3\}$, where $vu_1u_2u_3v$ and $vw_1w_2w_3v$ are the two cycles in $C_4 \cdot C_4$.

Case 1: $x = y$. By Lemma 6, we may assume that $k = 2, 3$, or 4. If $x = y = u_1$, then G can be obtained from C_3 or C_5 by consecutively attaching a path of order 3 when $k = 2$ or 4, and G has a spanning subgraph that is obtained from two disjoint C_4s by attaching a path of order 2 when $k = 4$, so G is configurable by Lemma 5 and Lemma 7. Similarly, G is configurable if both x and y are u_3, w_1, or w_3. If $x = y = v_2$ and $k = 2$ or 4, then G can be obtained from C_3 or C_5 by consecutively attaching a path of order 3, so G is configurable by Lemma 5. If $x = y = u_2$ and $k = 3$, then we define a configuration on G as $f(v) = \{3, 4\}, f(w_1) = \{1, 3\}, f(w_2) = \{2, 5\}, f(w_3) = \{1, 4\}, f(u_1) = \{4, 5\}, f(u_2) = \{1, 2\}, f(u_3) = \{2, 5\}, f(v_1) = \{1, 3\}, f(v_2) = \{4, 5\}, and f(v_3) = \{2, 3\}$. Similarly, G is configurable if $x = y = w_2$. If $x = y = v$ and $k = 2$ or 4, then G can be obtained from C_3 or C_5 by consecutively attaching a path of order 3. If $x = y = v$ and $k = 3$, then we define a configuration by $f(v) = \{1, 2\}, f(u_1) = \{1, 3\}, f(u_2) = \{4, 5\}, f(u_3) = \{2, 3\}, f(v_1) = \{1, 4\}, f(v_2) = \{3, 5\}, f(v_3) = \{2, 4\}, f(w_1) = \{1, 5\}, f(w_2) = \{3, 4\}, and f(w_3) = \{2, 5\}$.

Case 2: $x \neq y$. By Lemma 6, we may assume that $k = 0, 1, 2$. When $k = 0$, G is obtained by adding an edge xy to $C_4 \cdot C_4$, and it is easy to show that G is configurable. When $k = 1$, $x = v$, and $y = u_2$, then define a configuration on G by $f(v) = \{1, 2\}, f(u_1) = \{4, 5\}, f(u_2) = \{3, 4\}, f(u_3) = \{1, 5\}, f(v_1) = \{2, 5\}, f(w_1) = \{1, 3\}, f(w_2) = \{4, 5\}$, and $f(w_3) = \{2, 3\}$. Similarly, G is configurable if $k = 1, x = w_1$, and $y = w_3$. When $k = 1$ and x and y are not the case mentioned above, G has a spanning subgraph that is C_8, or it can be obtained from either C_5 by attaching a path, two disjoint C_4s by adding an edge, or C_5 by attaching paths of order 1 or 2, so G is configurable by Lemma 5, Lemma 7, and Lemma 8.

Now, we assume that G obtained from $K_{2,3}$ by attaching a path $v_1 v_2 ... v_k$, where v_1 is adjacent to x and v_k is adjacent to y for some vertices x and y in $C_4 \cdot C_4$. We write $V(K_{2,3}) = \{u_1, u_2, w_1, w_2, w_3\}$ and $E(K_{2,3}) = \{u_i w_j : i = 1, 2, j = 1, 2, 3\}$.

Case 3: $x = y$. By Lemma 6, we may assume that $k = 2, 3, 4$. Then G has a spanning subgraph that is obtained from either C_3 or C_5 by attaching a $(3, 0)$-star, or $C_4 \cdot C_4$ by attaching a path, or a cycle by attaching a C_4, so G is configurable by Lemma 5, Lemma 7, Case 1, and Case 2.

Case 4: $x \neq y$. By Lemma 6, we may assume that $k = 0, 1, 2$. If $x = u_1$, $y = u_2$, and $k = 0$, then there is a configuration on G defined by $f(u_1) = \{1, 2\}, f(u_2) = \{3, 4\}, f(w_1) = f(w_2) = f(w_3) = \{1, 5\}$. For other cases, G contains a subgraph that is isomorphic to $K_{2,4}$ or C_6, or it can be obtained from either C_3 by attaching a path of order three, $C_4 \cdot C_4$ by adding an edge, C_5 or C_6 by attaching paths of order one or two, so G is configurable by Lemma 5, Lemma 8, Lemma 12, Case 1, and Case 2. □

3. A SPECIAL CASE

For a vertex v of a graph G, we denote the degree of v by $\deg_G(v)$.

Lemma 14. *For every graph G, there is an orientation of $E(G)$ such that each vertex v has indegree at least $\lfloor \deg_G(v)/2 \rfloor$.*

Proof. We proceed by induction on $|V(G)| + |E(G)|$. The lemma obviously holds for the null graph. If v is an isolated vertex of G, then the lemma follows by induction applied to $G\backslash v$. If there is a vertex v in G of degree 1, then, letting u be the unique neighbor of v, there is an orientation of $G \setminus uv$ such that the indegree of each vertex x is at least $\lfloor \deg_{G\setminus\{uv\}}(x)/2 \rfloor$ by the induction hypothesis, and then we can obtain a desired orientation of G by orienting the edge uv from v to u. So we may assume that G has minimum degree at least 2, and hence G contains a cycle $C = v_1 v_2 ... v_k v_1$. By the induction hypothesis, there is an orientation of $G \setminus E(C)$ such that the indegree of each vertex x is at least $\lfloor \deg_{G\setminus E(C)}/2 \rfloor$, and then we can obtain a desired orientation of G by orienting the edges of C to form a directed cycle. This completes the proof. □

Note that the proof in Lemma 14 gives a linear-time algorithm to find such an orientation.

Lemma 15. *Let H_1 and H_2 be graphs, let P be a path with at least one vertex, and let v_1 and v_2 be vertices of H_1 and H_2, respectively. Let G be the graph formed by taking the disjoint union of H_1, H_2, and P and identifying the first vertex of P with v_1 and the last vertex of P with v_2. Assume that f_1 and f_2 are functions mapping $V(H_1)$ and $V(H_2)$ to $[5]^2$, respectively, and that for $i = 1, 2$ the function f_i satisfies every vertex of H_i except possibly v_i. If $|\bigcup_{u\in N(v_1)} f_1(u)| \geq 4$ and $|\bigcup_{u\in N(v_2)} f_2(u)| \geq 3$, then G is configurable.*

Proof. Let f' be the function defined to be f_1 on H_1 and f_2 on H_2. Then f' is a configuration for G except possibly on v_1 and v_2 and P. Suppose $|V(P)| \leq 2$. Then we can permute the colors on f_2 so that v_1 and v_2 are satisfied, so we are done. If $|V(P)| = 3$, we may assume $f(v_1) = \{1, 2\}$ and v_1 is not missing a number except possibly 3 and $f(v_2) = \{4, 5\}$ and v_2 is not missing a number other than possibly 3 and a number c. Then we set $f(u) = \{c, 3\}$ where u is the middle vertex of P. If $|V(P)| = 4$, we apply Lemma 4. If $|V(P)| \geq 5$, we can reduce to one of the previous cases by applying Lemma 6. □

Lemma 16. *Let G be a graph and v a vertex of G. If G is isomorphic to C_4, then there exists a function $f : V(G) \to [5]^2$ such that v is satisfied and $|\bigcup_{u\in N[v]} f(u)| \geq 3$. If G is isomorphic to C_7, $C_4 \cdot C_4$ or $K_{2,3}$, then there exists a function $f : V(G) \to [5]^2$ such that v is satisfied and $|\bigcup_{u\in N[v]} f(u)| \geq 4$.*

Proof. This is easy to verify. □

We are now ready to prove an important special case of Theorem 1.

Lemma 17. *Let G be a connected graph of maximum degree at most 5 and of minimum degree at least 2 with no two vertices of degree at least 3 adjacent. If G is not C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$, then G is configurable.*

Proof. Let n be the order of G. Suppose that G is a minimum counterexample; that is, G is not configurable, but H is configurable for every graph H with $|V(H)| + |E(H)| < |V(G)| + |E(G)|$ that satisfies the conditions of the lemma.

We note first that we may assume G is 2-connected. Otherwise we apply Lemma 15, noting that each of the forbidden graphs except C_4 has the property that for every vertex v, it admits a function f that satisfies every vertex except v and $|\bigcup_{u\in N[v]} f(u)| = 4$ by Lemma 16. Since both graphs can not be C_4 (since $C_4 \cdot C_4$ is forbidden and two C_4s joined by a path are prevented by Lemma 7), we are done.

The proof of this lemma is organized as follows. We first prove structure properties of G in Claims 1–4. And the rest of the proof is dedicated to a construction of a configuration function of G. It will lead to a contradiction.

Claim 1. *G contains no C_4s.*

Proof of Claim 1. Suppose there is a cycle $C = v_1v_2v_3v_4v_1$ of four vertices in G. If there is only one vertex, say v_1, in C of degree at least 3 in G, then it is a cut-vertex that is impossible.

Hence there are two vertices in C of degree at least 3. We may assume that the two vertices are v_1 and v_3. Let $G' = G \setminus \{v_2\}$. If G' is configurable, then there is a configuration f on G', and we can extend f to G by assigning $f(v_2) = f(v_4)$, contradicting the assumption that G is not configurable. Note that G' is a connected graph of maximum degree at most 5 and of minimum degree at least 2 with no two vertices of degree at least 3 adjacent. Since $|V(G')| + |E(G')| < |V(G)| + |E(G)|$, G' is isomorphic to C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$. If G' is isomorphic to C_4, then G is isomorphic to $K_{2,3}$. If G' is isomorphic to C_7, then G is isomorphic to a graph obtained from C_4 by adding a path of length five, so G is configurable by Lemma 11. If G' is isomorphic to $K_{2,3}$, then G is $K_{2,4}$, and it is configurable by Lemma 12. So G' is isomorphic to $C_4 \cdot C_4$. Since v_4 is a vertex of degree 2 and it is a common neighbor of v_1 and v_3, we have that either v_1 or v_3 is the vertex of degree 4 in $C_4 \cdot C_4$. So G can be obtained from adding a path of length 4 to $K_{2,3}$, so G is configurable by Lemma 13. ■

Claim 2. *If P is a path whose ends are of degree at least 3 in G and whose internal vertices are of degree 2 in G, then the number of internal vertices is at most 2.*

Proof of Claim 2. If the number of internal vertices of P is at least four, then consider the graph H that is obtained from G by replacing three consecutive degree 2 vertices in P by an edge. If H is configurable, G is also configurable by Lemma 6. So H is C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$. But in this case, G can be obtained from C_4 by attaching a path of order at least 3, so G is configurable by Lemma 11. If the number of internal vertices of P is three, then let H' be the graph obtained from P by deleting all internal vertices of P. Again, G is configurable by Lemma 5 if H' is configurable. So H' is C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$. However, G is configurable by Lemma 11 and Lemma 13 in this case. ■

Claim 3. *There are no induced (α, β)-stars S in G, where $\alpha + \beta \geq 3$, and $\alpha + 3\beta \leq 9$ or $(\alpha, \beta) = (1, 3)$, such that $G \setminus S$ has minimum degree at least 2.*

Proof of Claim 3. Suppose there is an induced (α, β)-star S, where $\alpha + \beta \geq 3$, and $\alpha + 3\beta \leq 9$ or $(\alpha, \beta) = (1, 3)$, such that $G \setminus S$ has minimum degree at least 2. Subject to this constraint, assume that $\alpha + \beta$ is as small as possible. Let $G' = G \setminus S$, and $M_1, M_2, ..., M_k$ be components of G'. If every component of G' is configurable, then G is configurable by Lemma 5. So there is a component of G' that is not configurable, and hence this component is isomorphic to C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$ by the minimality of G. But G contains no C_4s by Claim 1, so the component is isomorphic to C_7. Without loss of generality, we may assume that M_1 is isomorphic to C_7 and write $M_1 = v_1v_2...v_7v_1$.

If M_1 contains exactly one vertex of degree at least 3 in G, then G is configurable by Lemma 7, a contradiction. If M_1 contains exactly two vertices of degree at least 3 in G, then there is a path of length at least 4 whose ends are of degree at least 3 in G and whose internal vertices are of degree 2 in G, contradicting Claim 2. Hence there are three vertices in M_1 of degree at least 3 in G, and we may assume that they are v_1, v_3, and v_5. Furthermore, if all v_1, v_3, and v_5 have degree at least 4 in G, then $\alpha + \beta \geq 6$. Since $\alpha + 3\beta \leq 9$, we have that $\beta \leq 1$ and G contains a C_4, contradicting Claim 1. So at least one of v_1, v_3, and v_5, say x, has degree 3 in G. Note that there is an (α, β)-star with center x and $\alpha + \beta = 3$ such that the graph obtained from G by deleting this (α, β)-star is still of minimum degree at least 2, so S must also have that $\alpha + \beta = 3$ by the minimality of $\alpha + \beta$. So G' is C_7 as $\alpha + \beta = 3$. In other words, $(\alpha, \beta) = (0, 3)$, $(1, 2)$, $(2, 1)$, or $(3, 0)$.

If $(\alpha, \beta) = (3, 0)$, then G can be obtained from C_6 by attaching a $(2, 1)$-star, so G is configurable by Lemma 5. So this is not a $(3, 0)$-star. If $(\alpha, \beta) = (0, 3)$, then G is configurable since it can be obtained from C_8 by attaching a $(1, 2)$-star. If $(\alpha, \beta) = (1, 2)$, then G is configurable since G can be obtained from C_8 by attaching either a $(2, 1)$-star or $(3, 0)$-star. So $(\alpha, \beta) = (2, 1)$. Let $V(S) = \{a, b, c, d_1, d_2\}$ and $E(S) = \{ab, ac, ad_1, d_1d_2\}$. If d_2 is adjacent to v_1 or v_5, then G is configurable since it can be obtained from C_6 by attaching a $(1, 2)$-star. So d_2 is adjacent to v_3. Hence there is a configuration on G defined as $f(v_1) = \{1, 2\}$, $f(v_2) = \{4, 5\}$, $f(v_3) = \{1, 3\}$, $f(v_4) =$

$\{4, 5\}$, $f(v_5) = \{1, 2\}$, $f(v_6) = \{3, 4\}$, $f(v_7) = \{3, 5\}$, $f(a) = \{1, 3\}$, $f(b) = f(c) = \{4, 5\}$, $f(d_1) = \{2, 5\}$, and $f(d_2) = \{2, 4\}$. This proves Claim 3. ∎

Claim 4. *G contains no C_6 with exactly two vertices of degree at least 3 that are diagonally opposite on the cycle.*

Proof of Claim 4. Let $C = v_1 v_2 ... v_6 v_1$ be a cycle of order 6 with v_1 and v_4 the two vertices of degree at least 3 in G. Since G has no adjacent vertices whose degrees are at least 3, v_5 and v_6 have degree 2 in G. Let G' be the graph obtained by deleting v_5 and v_6 from G, so G' is a graph of minimum degree at least 2, maximum degree at most 5, and there are no adjacent vertices whose degrees are at least 3. If G' is not configurable, then G' is C_4, C_7, $C_4 \cdot C_4$ or $K_{2,3}$ by the minimality of G. However, G contains no C_4s, so G' is C_7 and it contains at most two vertices whose degrees in G are at least 3. Hence, there is a path of order at least 5 whose internal vertices are of degree 2, which contradicts to Claim 2. Consequently, G' is configurable and there is a configuration f on G', and we can extend f to G by defining $f(v_5) = f(v_3)$ and $f(v_6) = f(v_2)$. ∎

We now construct a configuration on G. Construct a graph H as follows: the vertices of H are the vertices of degree at least 3 in G, and xy is an edge in H if x and y have a common neighbor in G.

Claim 5. *The maximum degree of H is at most 2.*

Proof of Claim 5. Suppose there is a vertex x of degree at least 3 in H. Let $x_1, x_2, ..., x_k$ be the vertices of degree at least 3 such that there exist x-x_i paths of length 2 or 3. Then the internal vertices of those x-x_i paths together with x form an (α, β)-star S with $\alpha \geq 3$. On the other hand, $\alpha + \beta$ is at most 5 since G is of maximum degree at most 5. So S is an (α, β)-star with $\alpha + 3\beta \leq 9$. By Claim 3, $G \setminus S$ is not of minimum degree at least 2. So the degree of x_i in $G \setminus S$ is at most one, for some $i = 1, 2, ..., k$. Since G contains no C_4s and C_6 with exactly two diagonal vertices of degree at least 3 in G, the degree of x_i is exactly 3. So there is an (α', β')-star S' centered at x_i with $\alpha' + 3\beta' \leq 9$ such that $G \setminus S'$ is of minimum degree 2 since $\alpha \geq 3$, which contradicts Claim 3. Hence, the maximum degree of H is at most 2. ∎

By Claim 5, H is a disjoint union of isolated vertices, paths, and cycles. Let H^2 be the graph obtained by adding edges xy to H for each pair of two vertices x and y that have distance exactly two between them in H, and then deleting multiple edges and loops. So H^2 has maximum degree at most 4. Let H' be the graph that is obtained by deleting an edge that is in H^2 but not in H from each component of H^2 isomorphic to K_5. Hence, H' is 4-colorable by Brooks' theorem. Let $c : V(H') \to \{1, 2, 3, 4\}$ be a proper 4-coloring of H' such that $c(v) = 1$ for each isolated vertex v in H. Note that H^2 contains a component that is isomorphic to K_5 if and only if the component in H is isomorphic to C_5.

Define a function $f : V(H) \to [5]^2$ as $f(v) = \{c(v), 5\}$ for every vertex v in H. Let U be the set of vertices u such that u is a common neighbor of two vertices of degree at least 3 in G. Since no two vertices of degree at least 3 are adjacent, every vertex in U is of degree 2 in G. Now, we shall extend f to $V(H) \cup U$ by defining $f(u) = \{1, 2, 3, 4, 5\} \setminus (f(x) \cup f(y))$ for each vertex u in U, where x and y are the two neighbors of u in G. Note that if x and y are the two neighbors of a vertex u in U, then $c(x) \neq c(y)$ since H' contains all edges in H, so $|f(x) \cup f(y)| = 3$, and f is well-defined on $V(H) \cup U$. It is clear that $\bigcup_{w \in N[u]} f(u) = \{1, 2, 3, 4, 5\}$ for each $u \in U$. Furthermore, if v is a vertex with degree at least 2 in H, and v is not in a component of H isomorphic to C_5, then neighbors of v in

H receive different colors under c, so u is satisfied. Similarly, for each component of H that is isomorphic to C_5, there is a vertex w such that $|\bigcup_{u\in N[w]\cap(V(H)\cup U)} f(u)| = 4$ and each other vertex is satisfied.

Let W be the set of vertices w that are not satisfied. So each vertex in W is either an isolated vertex in H, an end of a maximal path in H, or a vertex in a component of H that is isomorphic to C_5. Let $X = \{w \in W : w \text{ is an isolated vertex in } H\}$, and let Y be the set $W \setminus X$. Notice that $|\bigcup_{u\in N[w]\cap(V(H)\cup U)} f(u)| = 4$ when w is in Y. Now, construct a graph L, where $V(L)$ is equal to $V(H)$, and two vertices x and y in L are adjacent if there is a x-y path of length 3 in G. Note that since no vertices of degree at least 3 are adjacent, the internal vertices of every x-y path of length 3 in G are of degree 2 for each $xy \in E(L)$.

Claim 6. *If w is in X, then the degree of w in L is at least 4. If w is in Y, then the degree of w in L is at least 2.*

Proof of Claim 6. Let w be a vertex in $X \cup Y$. Let $x_1, x_2, ..., x_k$ be vertices of degree at least 3 in G such that there are w-x_i paths in G of length two or three for each $i = 1, 2, ..., k$. Then the internal vertices of those w-x_i paths together with w form an (α, β)-star S.

Suppose $w \in X$. Then $\alpha = 0$ and there is at most one path between w and each x_i since otherwise we violate Claim 4. But then $G\backslash S$ has minimum degree 2, so by Claim 3, $\beta \geq 4$, so the degree of w in L is at least 4.

Suppose $w \in Y$ and that $\beta \leq 1$. If w was not in a C_5 in H, then $\alpha = 1$, so the degree of w is only 2. So we must have that w was in a C_5 in H, so $\alpha = 2$. Removing S must create a vertex of degree 1 by Claim 3, say x_1. So x_1 must have degree 3 and be part of a 5-cycle D in G with w. Since w is in a C_5 in H, G must have that x_1 has a path of length 2 to another vertex of degree at least 3 in G and that the graph H' obtained from G by removing D and the two degree 2 vertices that are adjacent to vertices of D is connected and of minimum degree 2. If H' is configurable, then by Lemma 10, G would be as well, so H' must be C_7, which is impossible since it has at least one degree 3 vertex since G has at least five degree 3 vertices since w is in a C_5 in H. ■

By Lemma 14, L then has an orientation in which each vertex of X has indegree at least 2 and every vertex in Y has indegree at least 1. We use this to extend f to satisfy every vertex in G. Each edge in L corresponds to a path of length 3, x, v_1, v_2, y in G (where x is the tail of the edge in L). For each of these paths, let a and b be two colors not in $\bigcup_{u\in N(x)} f(u)$ (if that many colors exist, otherwise arbitrarily add colors not in $f(x)$). Then assign $f(v_1) = (a, b)$ and $f(v_2)$ as given by Lemma 4.

Clearly at the end of this process, each vertex of degree 2 is satisfied. Each vertex not in X or Y was already satisfied. Each vertex in X was the tail of two edges in L, so it sees up to four new colors, and so is certainly satisfied. Each vertex in Y was only missing at most two colors, but was the tail of at least one edge in L, so it is now satisfied. □

4. MAIN THEOREM

We now prove Theorem 1, which we restate in equivalent form.

Theorem 18. *If G is a connected graph of minimum degree at least 2 with no induced subgraph isomorphic to $K_{1,6}$, and G is not isomorphic to a member of $\{C_4, C_7, C_4 \cdot C_4, K_{2,3}, G_i : 1 \leq i \leq 4\}$, then G is configurable.*

Proof. We first prove the theorem for graphs on at most six vertices. It is easy to see that the theorem holds if $|V(G)| \leq 4$, so we assume that $5 \leq |V(G)| \leq 6$. If G contains C_6, then C_6 is a spanning subgraph of G. Since C_6 is configurable, G is configurable. So we may assume that G does not contain C_6. If G contains C_5, then G contains a spanning subgraph that is obtained from C_5 by attaching a path on one vertex. Since G does not contain C_6, G is configurable by Lemma 9. Hence, we may assume that the longest cycle in G has length at most 4.

Assume that G contains C_4. Since $|V(G)| \leq 6$, G is 2-edge-connected. So G contains a spanning subgraph that can be obtained from C_4 by consecutively attaching paths. If the first path we attached contains two vertices, then since G has no cycle of length greater than 4, G contains a spanning subgraph that can be obtained from a triangle by attaching a path on three vertices and hence is configurable by Lemma 11. If the first path we attached has only one vertex, then since G does not contain C_5, G contains a spanning subgraph that can be obtained from a triangle by attaching two paths on one vertex to different vertices or from $K_{2,3}$ by attaching a path on one vertex, so we are done by Lemmas 8 and 13.

Therefore, we may assume that every cycle in G is a triangle. If G is 2-edge-connected, then G can be obtained from C_3 by attaching a path on two vertices and hence is configurable by Lemma 8. If G is not 2-edge-connected, then G contains two disjoint triangles as a spanning subgraph, and hence G is configurable. This proves that the theorem holds for graphs on at most six vertices.

We now proceed by induction on $|V(G)| + |E(G)|$. We have shown the theorem holds for graphs on at most six vertices, so we may assume that the order of G is at least 7.

Suppose there is a vertex v of degree 2 in G such that v is in a $C_4 = vabcv$ with degree of b also two. Note that G_i contains a spanning cycle of length 7 for $1 \leq i \leq 4$. Suppose that the degree of a is also 2. If c is not of degree 3, then G is obtained by attaching a path on three vertices to a configurable graph or an exceptional graph, so G is configurable by Lemmas 5, 11, and 13. If c is of degree 3, then G is obtained from a C_4 and a graph by attaching a path, where the ends of the path are adjacent to vertices in different components. Then G is configurable by Lemmas 7 and 15. So we may assume that a and c have degree at least 3.

So we consider $G \setminus v$. If it has a configuration f, then G is configurable since we may extend f to $V(G)$ by assigning $f(v) = f(b)$. As the order of G is at least 7, $G \setminus v$ is not configurable only if $G \setminus v$ is $C_4 \cdot C_4$ or contains a spanning cycle of length 7. However, it is not hard to see that if $G \setminus v$ is $C_4 \cdot C_4$ or contains a spanning cycle of length 7, then G contains a spanning subgraph that can be obtained either from $C_4 \cdot C_4$ by attaching a path on one vertex or from C_4 by attaching a path on four vertices, so G is configurable by Lemmas 11 and 13. Hence we may assume that no 4-cycle has two vertices of degree 2 opposite one another.

Suppose there were three vertices x, y, and z in G such that x, y, and z form a triangle in G and the degrees of y and z were exactly 2. Assume that x is not of degree 3. By the induction hypothesis, Lemma 8 and Lemma 13, G is configurable if $G \setminus \{y, z\}$ is not C_4 or contains C_7 as a spanning subgraph. But if $G \setminus \{y, z\}$ is C_4 or contains C_7 as a spanning subgraph, then G contains a spanning subgraph that can be obtained from C_3 by attaching a path with order at least 3, so G is still configurable by Lemma 11. Similarly, if x is of degree 3, then G is configurable by Lemma 15. Hence, we may assume that G has no triangles with two vertices of degree 3.

Let G' be a spanning subgraph of G such that the minimum degree of G' is at least 2 and G' satisfies the following:

1. $|E(G')|$ is as small as possible;
2. Subject to that, the number of triangles in G' is as small as possible; and
3. Subject to that, the number of components in G' that are isomorphic to $C_4 \cdot C_4$ or $K_{2,3}$ is as small as possible.

We shall prove the following claim. Note that by the minimality of $E(G')$, there are no two vertices of degree at least 3 adjacent to one another.

Claim 1. *The maximum degree of G' is at most 5.*

Proof of Claim 1. Suppose that there is a vertex v of degree at least 6 in G'. As G is $K_{1,6}$-free, there are two vertices x and y adjacent to v in G' with x adjacent to y in G. Since the degree of v is at least 3, x and y must have degree 2 in G'. If $xy \notin E(G')$, then the graph obtained by deleting xv and yv from G' and then adding xy into G' is still a spanning subgraph of G with minimum degree at least 2, but it has fewer edges. So $xy \in E(G')$, in other words, v, x, and y form a triangle in G'. Since x, y, and v form a triangle in G and the degree of v is at least 3, at least one of x and y has degree at least 3 in G. We may assume that the degree of x in G is at least 3, and u is a neighbor of x in G other than y and v. As xy and $vx \in E(G')$ and the degree of x is 2 in G', $xu \notin E(G')$. So the graph obtained by deleting xv and adding xu has the same number of edges but it has fewer triangles than G', a contradiction. ■

Since every component of G' is a connected graph of minimum degree at least 2 and of maximum degree at most 5, and no vertices of degree at least 3 in G' are adjacent to one another, every component of G' is configurable except those that are isomorphic to $C_4, C_7, C_4 \cdot C_4$, or $K_{2,3}$ by Lemma 17. Also, it follows by a simple case checking that if a graph not containing C_7 as a spanning subgraph contains $C_4, C_4 \cdot C_4$ or $K_{2,3}$ as a spanning subgraph but not as an induced subgraph, then it is also configurable.

Now, we show that G is configurable. If $|V(G)| = 7$ but G is not configurable, then G contains C_7 as a spanning subgraph. We denote the C_7 by $v_0v_2...v_6$. If there exists i with $0 \leq i \leq 6$ such that v_iv_{i+2} is an edge, where the index is computed modulo seven, then G contains a spanning subgraph that can be obtained from C_3 by adding a path on four vertices, so G is configurable by Lemma 11. Since G is not C_7 or G_1, G contains at least nine edges. If there exists i with $0 \leq i \leq 6$ such that v_iv_{i+3} and $v_{i+1}v_{i+5}$ are edges of G, then G is configurable by Lemma 9. So G contains G_2 or G_3 as a subgraph but not an induced subgraph since G is not G_2 or G_3. In addition, adding an edge to G_2 or G_3 makes it configurable unless it creates G_4. But adding an edge to G_4 makes it configurable. This proves that G is configurable if G contains at most seven vertices. So we may assume that G has at least eight vertices.

Let H be a maximal configurable subgraph of G induced by a union of components of G'. Suppose that H is empty. Since G contains at least eight vertices, G' contains at least two components. Let H_1 and H_2 be two components of G' adjacent in G and v_i be a vertex of H_i adjacent in G to H_{3-i} for $i = 1, 2$. By Lemma 16, for each $C_4, C_7, C_4 \cdot C_4$, and $K_{2,3}$, and for each of its vertices v, there exists a function f mapping the vertices to $[5]^2$ satisfying every vertex except possibly v, and v is missing at most two colors. Let f_1, f_2 be such a function defined on $V(H_1)$ and $V(H_2)$, respectively, such that v_1 and v_2 are the only vertices missing some colors. Therefore, we can permute the colors in f_1

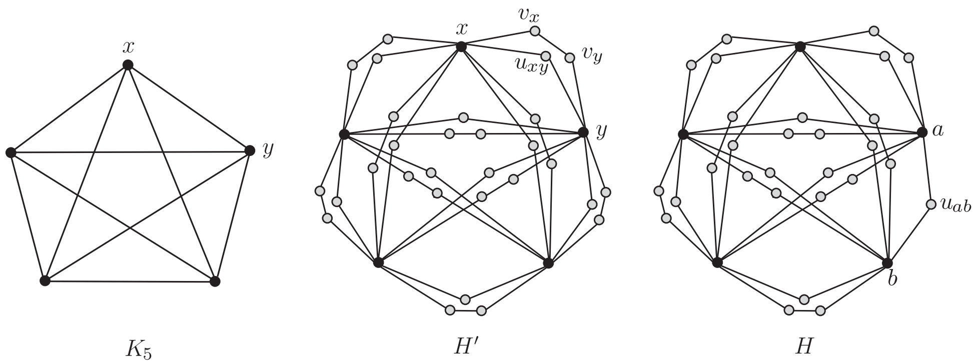

FIGURE 2. A complete graph K_5. H' is obtained by replacing every edge $xy \in E(K_5)$, by disjoint paths $xu_{xy}y$ and xv_xv_yy. H is obtained from H', by deleting v_a and v_b, from two distinct vertices a and b. Note that every vertex of H that belongs to the original K_5 has degree 8, except a and b that have degree 7.

and f_2 such that $f_i(v_i)$ contains the colors which v_{3-i} missed for $i = 1, 2$. This proves that the subgraph of G induced by $V(H_1) \cup V(H_2)$ is configurable, so H is not empty.

If $H \neq G$, then let C be a component of G' disjoint from H but adjacent in G to H. By Lemma 16, for every $v \in V(C)$, there exists a function f mapping the vertices to $[5]^2$ satisfying every vertex except possibly v, and v is missing at most two colors. Therefore, the subgraph of G induced by $V(H) \cup V(C)$ is configurable by Lemma 15, contradicting the maximality of H. This proves that $H = G$ and G is configurable. □

Note that our proof gives a polynomial-time algorithm to find a configuration of an n-vertex graph G if G is a $K_{1,6}$-free graph of minimum degree at least 2, and no component of G is isomorphic to C_4, C_7, $C_4 \cdot C_4$, or $K_{2,3}$.

Now we shall show that the hypothesis that G be $K_{1,6}$-free cannot be replaced by assuming that G be $K_{1,9}$-free. We do so by exhibiting infinitely many examples that contain no induced $K_{1,9}$ but are not configurable. Let H' be the graph obtained from K_5 by replacing each edge xy by two internally disjoint paths $xu_{xy}y$ and xv_xv_yy, and H be the graph obtained from H' by deleting v_a and v_b, where a and b are two distinct vertices in the original K_5. So the maximum degree of H is 8, and there are exactly two vertices that have degree 7. See Figure 2. Suppose that H is configurable and f is a configuration on H. If x and y are distinct vertices in the original K_5, then $f(x) \neq f(y)$ for otherwise $\bigcup_{z \in N[u_{xy}]} f(z) \neq \{1, 2, 3, 4, 5\}$, and $f(x) \cap f(y)$ is nonempty for otherwise $\bigcup_{z \in N[v_x]} f(z)$ or $\bigcup_{z \in N[v_y]} f(z)$ is not $\{1, 2, 3, 4, 5\}$. But if S is a subset of $[5]^2$ such that every two members of S have a nonempty intersection, then the size of S is at most 4, so $f(a) = f(b)$. However, this implies $\bigcup_{w \in N[u_{ab}]} f(w) \neq \{1, 2, 3, 4, 5\}$, a contradiction. Hence, H is not configurable. For any positive integer k, let $H_1, H_2, ..., H_k$ be graphs, where each of them is isomorphic to H, and a_i, b_i are the two vertices of degree 7 of H_i for each $i = 1, 2, ..., k$. Let G be the graph obtained from $H_1 \cup H_2 \cup \cdots \cup H_k$ by adding the edges b_ia_{i+1} for all $i = 1, 2, ..., k-1$ and b_ka_1, so G is of maximum degree 8 but not configurable.

On the other hand, one might ask whether we can get rid of the assumption about forbidden subgraphs by assuming the minimum degree is large. However, the following examples show that for every integer $k > 0$, there is a graph G with minimum degree k that is not configurable. Let $n = 10k - 9$, let B be a set of size n, and let A be the set of all k-element subsets of B. Let G be the graph with vertex-set $A \cup B$ in which a vertex $S \in A$ is adjacent to each of its elements.

By the pigeon-hole principle, there is a set S in A such that $f(b)$ are the same for all $b \in S$. But this implies that $|\bigcup_{v\in N[S]} f(v)| \leq 4$, a contradiction. So G is not configurable.

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